Last time: Found RNM $\to \tilde{P}$

(1) Know if sec pays V_T at time T

& sec is rep then

AFP at time t $= V_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$

$D_t = \exp\left(-\int_0^t R_s \, ds\right)$ $R_t =$ int Rate.

(2) Under P $\longrightarrow$ $dS_t = \bar{\alpha}_t S_t \, dt + \tilde{\sigma}_t S_t \, dW_t$

(3) $$Z_T = \exp\left(- \int_0^T \theta_t \, dW_t - \frac{1}{2} \int_0^T \theta_t^2 \, dt \right)$$

$$\theta_t = \frac{\alpha_t - R_t}{\sigma_t} \quad (\text{MPR})$$

$$\boxed{d\tilde{P} = Z_T \, dP}$$

(4) $d\tilde{W} = \theta_t \, dt + dW.$ $\tilde{W}$ is a BM under $\tilde{P}$ (up to time T)

$$\underline{\text{AND}} \quad \boxed{dS_t = R_t\,dt + \sigma_t\,d\tilde{W}}$$

Use this to compute Prices.

__Eg 1:__ Black Scholes Model.

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW \qquad \alpha, \sigma \text{ constants.}$$

Interest rate r (constant).

Security pays $g(S_T)$ at time T. Find AFP for $t \leq T$.

(assume sec is replicable (FDV))

$$D_t = \exp\left(-\int_0^t R_s\, ds\right) = e^{-rt}$$

Sol: AFP at time $t = V_t$

$$V_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$$

$$= e^{-r(T-t)} \tilde{E}_t V_T = e^{-rT} \tilde{E}_t\, g(S_T)$$

Know $dS_t = r S_t\, dt + \sigma S_t\, d\tilde{W}_t$

$\Rightarrow$ Under $\tilde{P}$, S is $GBM(r, \sigma)$

(Under P, S is a $GBM(r, \sigma)$)

$$\Rightarrow \quad S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma \tilde{W}_t \right)$$

$$S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma \tilde{W}_T \right)$$

$$\boxed{\Rightarrow \quad S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right) \tau + \sigma \left(\tilde{W}_T - \tilde{W}_t \right) \right)}$$

Hence $V_t = e^{-r\tau} \tilde{E}_t \, g(S_T)$

$$= e^{-r\tau} \tilde{E}_t \, g\Big(\underbrace{S_t}_{\mathcal{F}_t \text{ meas.}} \exp\Big(\left(r - \frac{\sigma^2}{2}\right) \tau + \sigma \underbrace{\left(\tilde{W}_T - \tilde{W}_t \right)}_{\text{indep of } \mathcal{F}_t} \Big) \Big)$$

By indep lemma:

Let $h(x) = \tilde{E}\, g\left(x \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\left(\frac{\tilde{W}_T - \tilde{W}_t}{\sqrt{\tau}}\right)\right)\right)$

Then $V_t = e^{-r\tau} h(S_t)$

Lazy Stat $\Rightarrow$ $h(x) = \displaystyle\int_{-\infty}^{\infty} g\left(x \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y\right)\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$

$\Rightarrow V_t = e^{-r\tau} h(S_t) = f(t, S_t)$

where $f(t, x) = e^{-r\tau} h(x)$

$$\boxed{\Rightarrow f(t,x) = e^{-r\tau}\int_{-\infty}^{\infty} g\left(x\exp\left[\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,y\right]\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy}$$

QED.

Eg 2: Put $g(x) = (x-K)^+$ (Payoff of a Eur call).

$c(t,x) =$ AFP if spot price $= x$.

Compute & simplify c:

$$c(t,x) = e^{-r\tau}\int_{-\infty}^{\infty}\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,y\right) - K\right)^+ \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy.$$

... $(*)$

Find y where $[\quad] \geq 0$

$$\iff x \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y\right) \geq K$$

$$\iff \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y \geq \ln\left(\frac{K}{x}\right) = -\ln\left(\frac{x}{K}\right)$$

$$\iff y \geq \frac{1}{\sigma\sqrt{\tau}}\left(-\ln\left(\frac{x}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau\right)$$

$$\iff y \geq -d_- \quad \text{where } d_- = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right]$$

$$\therefore\ (*) \Rightarrow c(t,x) = e^{-r\tau}\int_{-d_-}^{\infty}\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}\,y\right)-K\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

$$= e^{-r\tau}\int_{-d_-}^{\infty}(\ \ ) - e^{-r\tau}K\underbrace{\int_{-d_-}^{\infty}\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy}_{N(d_-)}$$

$$= (\ \ - Ke^{-r\tau}N(d_-)$$

$$= e^{-r\tau}\int_{-d_-}^{\infty}x\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}\,y-\frac{y^2}{2}\right)\frac{dy}{\sqrt{2\pi}} - Ke^{-r\tau}N(d_-)$$

$$= x \int_{-d_-}^{\infty} \exp\left(-\frac{\sigma^2\tau}{2} + \sigma\sqrt{\tau}\, y - \frac{y^2}{2}\right)\frac{dy}{\sqrt{2\pi}} - K e^{-r\tau} N(d_-)$$

$$= x \int_{-d_-}^{\infty} \exp\left(-\frac{1}{2}\big(\underbrace{y-\sigma\sqrt{\tau}}_{z}\big)^2\right)\frac{dy}{\sqrt{2\pi}} - K e^{-r\tau} N(d_-)$$

$$= x \int_{-(d_- + \sigma\sqrt{\tau})}^{\infty} \exp\left(-\frac{z^2}{2}\right)\frac{dz}{\sqrt{2\pi}} - K e^{-r\tau} N(d_-)$$

$$= x N(d_+) - K e^{-r\tau} N(d_-).$$

QED

$$\left(d_+ = d_- + \sigma\sqrt{\tau}\right)$$