LECTURE NOTES ON CONTINUOUS TIME FINANCE SPRING 2024

GAUTAM IYER

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213.

E-mail address: gautam@math.cmu.edu.

Contents

1. **Preface.**

These are the notes I used while teaching an undergraduate course on *Continuous time finance* at Carnegie Mellon University in Fall 2022. I filled in all proofs and details by hand during lectures, and these notes only contain statements and definitions. A PDF of these notes is on the class website, and the source code is available on git.

If you find these notes useful, you may modify them as needed to suit your purposes. In this case, please consider contributing your changes back here.

2. **Introduction.**

(1) Binomial model: Trade at discrete time intervals (370).
(2) Suppose now we can trade *continuously in time.*
(3) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(4) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$.
(5) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(6) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(7) *(Black-Scholes Formula)* The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

where $$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy.$$

(8) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

Remark 2.1. There's no explicit formula for the option price for fixed N in the Binomial model. But there's a "nice" explicit formula when $N \to \infty$.

3. **Central limit theorem (review).**

Definition 3.1. We say X is a *normally distributed* random variable with mean μ and variance σ^2 if the PDF of X is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big).$$

Remark 3.2. Notation: $X \sim \mathcal{N}(\mu, \sigma^2)$.

Remark 3.3. Normally distributed random variables are also called *Gaussian.*

Let $X_1, \dots, X_n$ be a sequence of i.i.d. random variables, with $\boldsymbol{E}X_n = 0$ and $\operatorname{Var} X_n = 1$. Let $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$.

Question 3.4. *How does S_n behave as $n \to \infty$.*

Theorem 3.5 (Law of large numbers)**.** $S_n/n \to 0$ *as* $n \to \infty$.

Remark 3.6. Easy check: Compute $\operatorname{Var}(S_n/n)$.

Theorem 3.7 (Central limit theorem)**.** $S_n/\sqrt{n} \to \mathcal{N}(0,1)$. *That is, for any bounded continuous function f,*

$$\boldsymbol{E}f\Big(\frac{S_n}{\sqrt{n}}\Big) = \boldsymbol{E}f(\mathcal{N}(0,1)).$$

Let X be a random variable.

Definition 3.8. The *characteristic function* of X is defined by $\varphi_X(\lambda) = \boldsymbol{E}e^{i\lambda X}$.

Definition 3.9. The *moment generating function* (MGF) of X is defined by $M_X(\lambda) = \boldsymbol{E} e^{\lambda X}$.

Example 3.10. If $X \sim N(0,1)$ then $\varphi_X(\lambda) = e^{-\lambda^2/2}$, and $M_X(\lambda) = e^{\lambda^2/2}$.

Theorem 3.11. $\boldsymbol{E} X^n = (-i)^n \varphi_X^{(n)}(0) = M_X^{(n)}(0)$. *In particular,* $\boldsymbol{E} X = -i\varphi'_X(0) = M'_X(0)$, *and* $\boldsymbol{E} X^2 = -\varphi''_X(0) = M''_X(0)$.

Remark 3.12. Here $f^{(n)}(0)$ denotes the n^{th} derivative of f at 0.

Let X, Y be two random variables.

Theorem 3.13. *The following are equivalent.*

(1) *X and Y have the same distribution (PDF)*
(2) *X and Y have the same CDF.*
(3) *X and Y have the same characteristic function.*
(4) *X and Y have the same moment generating function.*

Theorem 3.14. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $\varphi_{X_n} \to \varphi_X$ pointwise.*

Theorem 3.15. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $M_{X_n} \to M_X$ pointwise.*

Remark 3.16. The proofs of Theorem 3.13–3.15 are beyond the scope of this course; we will use them without proof.

Proof of Theorem 3.7.

4. Stochastic Processes.

4.1. Brownian motion.

- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and $\text{Range}(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t-n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Repeat more frequently: Flip a coin every ε seconds, and take a step of size $\sqrt{\varepsilon}$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\text{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

Definition 4.1 (Brownian motion)**.** The process W above is called a Brownian motion.

▷ Named after Robert Brown (a botanist).
▷ Definition is intuitive, but not as convenient to work with.

- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \displaystyle\sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$.
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 4.2. A (standard) Brownian motion is a *continuous process* such that:

(1) $W_0 = 0$, $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Remark 4.3. We will define $\mathcal{F}_s$ shortly. Intuitively, $\mathcal{F}_s$ is the set of all events that are "known" at time s.

4.2. **Sample space, measure, and filtration.**

- Discrete time: Sample space $\Omega = (\omega_1, \dots, \omega_N)$.
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0,\infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.
- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P} \colon \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\Big(\bigcup_1^\infty A_n\Big) = \lim_{n\to\infty} \boldsymbol{P}(A_n)$.
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.
- Expectation is a *Lebesgue Integral*: Notation $\boldsymbol{E}X = \int_\Omega X \, d\boldsymbol{P} = \int_\Omega X(\omega) d\boldsymbol{P}(\omega)$.
  - ▷ No simple formula.
  - ▷ If $X = \sum a_i \mathbf{1}_{A_i}$, then $\boldsymbol{E}X = \sum a_i \boldsymbol{P}(A_i)$.
  - ▷ $\mathbf{1}_A$ is the *indicator function* of A: $\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

Proposition 4.4 (Useful properties of expectation).

(1) (Linearity) *$\alpha, \beta \in \mathbb{R}$, X, Y random variables, $\boldsymbol{E}(\alpha X + \beta Y) = \alpha \boldsymbol{E}X + \beta \boldsymbol{E}Y$.*

(2) (Positivity) *If $X \geqslant 0$ then $\boldsymbol{E}X \geqslant 0$. If $X \geqslant 0$ and $\boldsymbol{E}X = 0$ then $X = 0$ almost surely.*

(3) (Layer Cake) *If $X \geqslant 0$, then $\boldsymbol{E}X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt$.*

(4) More generally, if φ is increasing, $\varphi(0) = 0$ then

$$\boldsymbol{E}\varphi(X) = \int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt\,.$$

(5) (Unconscious Statistician Formula) *If PDF of X is p, then*

$$\boldsymbol{E}f(X) = \int_{-\infty}^\infty f(x) p(x)\, dx\,.$$

- Filtrations:
  - ▷ Discrete time: $\mathcal{F}_n$ = events described using the first n coin tosses.
  - ▷ Coin tosses doesn't translate well to continuous time.
  - ▷ Discrete time try #2: $\mathcal{F}_n$ = events described using the *trajectory* of the SRW up to time n.
  - ▷ Continuous time: $\mathcal{F}_t$ = events described using the *trajectory* of the *Brownian motion* up to time t.
  - ▷ If $t_i \leqslant t$, $A_i \subseteq \mathbb{R}$ then $\{W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\} \in \mathcal{F}_t$. (Need all $t_i \leqslant t$!)
  - ▷ As before: if $s \leqslant t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$.
  - ▷ Discrete time: $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Continuous time: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \boldsymbol{P}(A) \in \{0,1\}\}$.

4.3. **Conditional expectation.**

- Notation $\boldsymbol{E}_t(X) = \boldsymbol{E}(X \mid \mathcal{F}_t)$ (read as conditional expectation of X given $\mathcal{F}_t$)
- No formula! But same intuition as discrete time.
- $\boldsymbol{E}_t X(\omega) =$ "average of X over $\Pi_t(\omega)$", where $\Pi_t(\omega) = \{\omega' \in \Omega \,|\, \omega'(s) = \omega(s)\ \forall s \leqslant t\}$.
- Mathematically problematic: $\boldsymbol{P}(\Pi_t(\omega)) = 0$ (but it still works out.)

Definition 4.5. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$

Remark 4.6. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 4.7 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \beta Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*

Remark 4.8. These properties are exactly the same as in discrete time.

Lemma 4.9 (Independence Lemma)**.** *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X, Y) = g(X), \qquad \text{where} \qquad g(x) = \boldsymbol{E} f(x, Y).$$

Remark 4.10. If p_Y is the PDF of Y, then $\boldsymbol{E}_t f(X, Y) = \displaystyle\int_{\mathbb{R}} f(X, y)\, p_Y(y)\, dy$.

Example 4.11. If X, Y are two independent standard normal random variables, find $\boldsymbol{E} e^{iXY}$.

4.4. **Martingales.**

Definition 4.12. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 4.13. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Proposition 4.14. *Brownian motion is a martingale.*

Proof.

Question 4.15. *Is W_t^2 a martingale? How about W_t^3?*

5. **Stochastic Integration**

5.1. **Motivation.**

- Hold b_t shares of a stock with price S_t.
- Only trade at times $P = \{0 = t_1 < \dots, t_n = T\}$

- Net gain/loss from changes in stock price: $\sum_{k=0}^{n-1} b_{t_k} \Delta_k S$, where $\Delta_k S = S_{t_{k+1}} - S_{t_k}$.
- Trade continuously in time. Expect net gain/loss to be $\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{t_k} \Delta_k S = \int_0^T b_t \, dS_t$.
  ▷ $\|P\| = \max_k (t_{k+1} - t_k)$.
  ▷ Riemann-Stieltjes integral: $\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{\xi_k} \Delta_k S = \int_0^T b_t \, dS_t$,
  ▷ The $\xi_k \in [t_k, t_{k+1}]$ can be chosen arbitrarily.
  ▷ Only works if the *first variation* of S is finite. False for most stochastic processes.

5.2. **First Variation.**

Definition 5.1. For any process X, define the *first variation* by

$$V_{[0,T]}(X) \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |\Delta_k X| \, . \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}| \, .$$

Remark 5.2. If $X(t)$ is a differentiable function of t then $V_{[0,T]} X < \infty$.

Proposition 5.3. $\boldsymbol{E} V_{[0,T]} W = \infty$

Remark 5.4. In fact, $V_{[0,T]} W = \infty$ almost surely. Brownian motion does not have finite first variation.

Remark 5.5. The Riemann-Stieltjes integral $\int_0^T b_t \, dW_t$ *does not exist.*

Proof of Proposition 5.3.

5.3. **Quadratic Variation.**

Definition 5.6. If M is a continuous time adapted process, define

$$[M, M]_T = \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} (\Delta_k M)^2 \, .$$

Proposition 5.7. *For continuous processes the following hold:*
  (1) Finite first variation implies the quadratic variation is 0
  (2) Finite (non-zero) quadratic variation implies the first variation is infinite.

Proposition 5.8. $[W, W]_T = T$ *almost surely.*

Remark 5.9. For use in the proof: $\operatorname{Var}(\mathcal{N}(0, \sigma^2)^2) = \boldsymbol{E}\mathcal{N}(0, \sigma^2)^4 - (\boldsymbol{E}\mathcal{N}(0, \sigma^2)^2)^2 = 2\sigma^4$.

Proof:.

Proposition 5.10. $W_t^2 - [W, W]_t$ *is a martingale.*

Theorem 5.11. *Let M be a continuous martingale.*
  (1) $\boldsymbol{E} M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M, M]_t < \infty$.
  (2) In this case $M_t^2 - [M, M]_t$ *is a continuous martingale.*

(3) Conversely, if $M_t^2 - A_t$ is a martingale for any continuous, increasing process A such that $A_0 = 0$, then we must have $A_t = [M, M]_t$.

Remark 5.12. If X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$.

Remark 5.13. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.

5.4. **Itô Integrals.**

- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times.
- $\|P\| = \max_i t_{i+1} - t_i$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$

Definition 5.14. The *Itô Integral* of D with respect to Brownian motion is defined by

$$I_T = \int_0^T D_t \, dW_t = \lim_{\|P\| \to 0} I_P(T) \,.$$

Remark 5.15. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\| \to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

Theorem 5.16. *If $\boldsymbol{E} \int_0^T D_t^2 \, dt < \infty$ a.s., then:*

(1) $I_T = \lim_{\|P\| \to 0} I_P(T)$ exists a.s., and $\boldsymbol{E} I(T)^2 < \infty$.

(2) The process I_T is a martingale: $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r \, dW_r = \int_0^s D_r \, dW_r = I_s$

(3) $[I, I]_T = \int_0^T D_t^2 \, dt$ a.s.

Remark 5.17. If we only had $\int_0^T D_t^2 \, dt < \infty$ a.s., then $I(T) = \lim_{\|P\| \to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

Corollary 5.18 (Itô isometry). $\boldsymbol{E}\Big(\int_0^T D_t \, dW_t\Big)^2 = \boldsymbol{E} \int_0^T D_t^2 \, dt$

Proof.

Intuition for Theorem 5.16 (2). Check $I_P(T)$ is a martingale.

Proposition 5.19. *If $\alpha, \tilde{\alpha} \in \mathbb{R}$, $D, \tilde{D}$ adapted processes*

$$\int_0^T (\alpha D_s + \tilde{\alpha} \tilde{D}_s) \, dW_s = \alpha \int_0^T D_s \, dW_s + \tilde{\alpha} \int_0^T \tilde{D}_s \, dW_s$$

Proposition 5.20. $\displaystyle\int_0^{T_1} D_s \, dW_s + \int_{T_1}^{T_2} D_s \, dW_s = \int_0^{T_2} D_s \, dW_s$

Question 5.21. *If $D \geqslant 0$, then must $\int_0^T D_t \, dW_t \geqslant 0$?*

5.5. **Semi-martingales and Itô Processes.**

Question 5.22. *What is* $\int_0^t W_s \, dW_s$*?*

Definition 5.23. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation.
- ▷ M is a martingale.

Definition 5.24. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s \, ds$, with $\int_0^t |b_s| \, ds < \infty$
- ▷ $M_t = \int_0^t \sigma_s \, dW_s$, with $\int_0^t |\sigma_s|^2 \, ds < \infty$

Remark 5.25. Short hand notation for Itô processes: $dX_t = b_t \, dt + \sigma_t \, dW_t$.

Remark 5.26. Expressing $X = X_0 + B + M$ (or $dX = b \, dt + \sigma \, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

Theorem 5.27 (Itô formula)**.** *If* $f \in C^{1,2}$*, then*

$$df(t, X_t) = \partial_t f(t, X_t) \, dt + \partial_x f(t, X_t) \, dX_t + \frac{1}{2} \partial_x^2 f(t, X_t) \, d[X, X]_t$$

Remark 5.28. This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

Proposition 5.29. *If* $X = X_0 + B + M$*, then* $[X, X] = [M, M]$*.*

Proposition 5.30 (Uniqueness)**.** *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$*, with:*

- ▷ B, C *bounded variation,* $B_0 = C_0 = 0$
- ▷ M, N *martingale,* $M_0 = N_0 = 0$*.*

Then $X_0 = Y_0$*,* $B = C$ *and* $M = N$*.*

Corollary 5.31. *Let* $dX_t = b_t \, dt + \sigma_t \, dW_t$ *with* $\boldsymbol{E} \int_0^t b_s \, ds < \infty$ *and* $\boldsymbol{E} \int_0^t \sigma_s^2 \, ds < \infty$*. Then* X *is a martingale if and only if* $b = 0$*.*

Definition 5.32. If $dX_t = b_t \, dt + \sigma_t \, dW_t$, then define

$$\int_0^T D_t \, dX_t = \int_0^T D_t b_t \, dt + \int_0^T D_t \sigma_t \, dW_t \, .$$

Remark 5.33. Note $\int_0^T D_t b_t \, dt$ is a *Riemann integral*, and $\int_0^T D_t \sigma_t \, dW_t$ is a *Itô integral.*

5.6. **Itô's formula.**

Remark 5.34. If f and X are differentiable, then

$$df(t, X_t) = \partial_t f(t, X_t) \, dt + \partial_x f(t, X_t) \, dX_t$$

Theorem (Itô's formula, Theorem 5.27)**.** *If* $f \in C^{1,2}$*, then*

$$df(t, X_t) = \partial_t f(t, X_t) \, dt + \partial_x f(t, X_t) \, dX_t + \frac{1}{2} \partial_x^2 f(t, X_t) \, d[X, X]_t$$

Remark 5.35. If $dX_t = b_t\,dt + \sigma_t\,dW_t$ then

$$df(t, X_t) = \Big(\partial_t f(t, X_t) + \partial_x f(t, X_t) b_t + \frac{1}{2}\partial_x^2 f(t, X_t)\sigma_t^2\Big)\,dt + \partial_x f(t, X_t)\sigma_t\,dW_t\,.$$

Intuition behind Itô's formula.

Example 5.36. Find the quadratic variation of W_t^2.

Example 5.37. Find $\displaystyle\int_0^t W_s\,dW_s$.

Example 5.38. Let $M_t = W_t$, and $N_t = W_t^2 - t$.
▷ We know M, N are martingales.
▷ Is MN a martingale?

Example 5.39. Let $X_t = t\sin(W_t)$. Let $Y_t = \int_0^t W_s\,dX_s$. Is Y a martingale? Is $X_t^2 - [X, X]_t$ a martingale?

Remark 5.40. If M is a martingale, then the Itô integral $N_t = \int_0^t D_s\,dM_s$ is also a martingale (provided $\boldsymbol{E}\int_0^t D_s^2 d[M, M]_s < \infty$). If X is not a martingale, however, the Itô integral $Y_t = \int_0^t D_s\,dX_s$ need not be a martingale.

Example 5.41. Say $dM_t = \sigma_t\,dW_t$. Show that $M^2 - [M, M]$ is a martingale.

Example 5.42. If $0 \leqslant r \leqslant s \leqslant t$, find $\boldsymbol{E}(W_s W_t)$ and $\boldsymbol{E}(W_r W_s W_t)$.

Example 5.43. Let $M_t = \displaystyle\int_0^t W_s\,dW_s$. Find a function f such that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \exp\Big(M_t - \int_0^t f(s, W_s)\,ds\Big)$$

is a martingale.

Theorem 5.44 (Lévy's criterion)**.** *If M is a continuous martingale with $M_0 = 0$ and $[M, M]_t = t$ then M is a standard Brownian motion.*

Proof:.

Remark 5.45. More generally, we we only know M is a continuous martingale, with $[M, M]_t = \alpha t$ for some $\alpha > 0$, then M is a Brownian motion. That is, for some $a, b \in \mathbb{R}$, the rescaled process $W = aM + b$ is a standard Brownian motion.

Remark 5.46. Requiring M is continuous is essential; the *compensated Poisson process* is a discontinuous martingale with $N_0 = 0$, $[N, N]_t = t$, but is not a standard Brownian motion.

6. Black Scholes Merton equation

6.1. Market setup and assumptions.

- Cash: simple interest rate r in a bank.
- Let Δt be small. $C_{n\,\Delta t}$ be cash in bank at time $n\,\Delta t$.
- Withdraw at time $n\,\Delta t$ and immediately re-deposit: $C_{(n+1)\Delta t} = (1 + r\,\Delta t)C_{n\Delta t}$.
- Set $t = n\Delta t$, send $\Delta t \to 0$: $\partial_t C = rC$ and $C_t = C_0 e^{rt}$.
- r is called the continuously compounded interest rate.

- Alternately: If a bank pays interest rate ρ after time T, then the equivalent continuously compounded interest rate is $r = \frac{1}{T}\ln(1+\rho)$.
- Stock price: $S_{t+\Delta t} = (1 + r\,\Delta t)S_t + \text{noise}$.
  - ▷ Variance of noise should be proportional to Δt.
  - ▷ Variance of noise should be proportional to S_t.
- $S_{t+\Delta t} - S_t = rS_t\,\Delta t + \sigma S_t(\Delta W_t)$.

Definition 6.1. A Geometric Brownian motion with parameters α, σ is defined by:

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t\,.$$

- α: Mean return rate (or percentage drift)
- σ: volatility (or percentage volatility)

Proposition 6.2. $S_t = S_0 \exp\Big(\Big(\alpha - \dfrac{\sigma^2}{2}\Big)t + \sigma W_t\Big)$

Market Assumptions.

- 1 stock, Price S_t, modelled by $\mathrm{GBM}(\alpha, \sigma)$.
- Money market: Continuously compounded interest rate r.
  - ▷ C_t = cash at time $t = C_0 e^{rt}$. (Or $\partial_t C_t = rC_t$.)
  - ▷ Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)

6.2. **The Black, Sholes, Merton equation.** Consider a security that pays $V_T = g(S_T)$ at maturity time T.

Theorem 6.3. *If the security can be replicated, and $f = f(t,x)$ is a function such that the wealth of the replicating portfolio is given by $X_t = f(t, S_t)$, then:*

$$\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0 \qquad x > 0,\ t < T\,, \tag{6.1}$$

$$f(t,0) = g(0)e^{-r(T-t)} \qquad t \leqslant T\,, \tag{6.2}$$

$$f(T,x) = g(x) \qquad x \geqslant 0\,. \tag{6.3}$$

Theorem 6.4. *Conversely, if f satisfies (6.1)–(6.3) then the security can be replicated, and $X_t = f(t, S_t)$ is the wealth of the replicating portfolio at any time $t \leqslant T$.*

Remark 6.5. Wealth of replicating portfolio equals the arbitrage free price.

Remark 6.6. $g(x) = (x-K)^+$ is a European call with strike K and maturity T.

Remark 6.7. $g(x) = (K-x)^+$ is a European put with strike K and maturity T.

Proposition 6.8. *A standard change of variables gives an explicit solution to (6.1)–(6.3):*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x\exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\,y\Big)\Big)\frac{e^{-y^2/2}dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{6.4}$$

Corollary 6.9. *For European calls, $g(x) = (x-K)^+$, and*

$$f(t,x) = c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{6.5}$$

where

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{6.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy\,, \tag{6.7}$$

is the CDF of a standard normal variable.

Remark 6.10. Equation (6.1) is called a *partial differential equation.* In order to have a unique solution it needs:

(1) A terminal condition (this is equation (6.3)),
(2) A boundary condition at $x = 0$ (this is equation (6.2)),
(3) A boundary condition at infinity (not discussed yet).
 - ▷ For put options, $g(x) = (K - x)^+$, the boundary condition at infinity is
$$\lim_{x \to \infty} f(t, x) = 0\,.$$
 - ▷ For call options, $g(x) = (x - K)^+$, the boundary condition at infinity is
$$\lim_{x \to \infty} \left[f(t, x) - (x - Ke^{-r(T-t)}) \right] = 0$$
That is, $f(t, x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty.$

Definition 6.11. If X_t is the wealth of a self-financing portfolio then

$$dX_t = \Delta_t\, dS_t + r(X_t - \Delta_t S_t)\, dt$$

for some adapted process Δ_t (called the trading strategy).

Proof of Theorem 6.3.
Proof of Theorem 6.4.
Proof of Theorem 6.4 (without discounting).

Remark 6.12. The arbitrage free price does not depend on the mean return rate!

Question 6.13. *Consider a European call with maturity T and strike K. The payoff is $V_T = (S_T - K)^+$. Our proof shows that the arbitrage free price at time $t \leqslant T$ is given by $V_t = c(t, S_t)$, where c is defined by (6.5). The proof uses Itô's formula, which requires c to be twice differentiable in x; but this is clearly false at $t = T$. Is the proof still correct?*

Proposition 6.14 (Put call parity)**.** *Consider a European put and European call with the same strike K and maturity T.*

▷ *$c(t, S_t) =$ AFP of call (given by (6.5))*
▷ *$p(t, S_t) =$ AFP of put.*

Then $c(t, x) - p(t, x) = x - Ke^{-r(T-t)}$, and hence $p(t, x) = Ke^{-r(T-t)} - x + c(t, x)$.

6.3. The Greeks.

Let $c(t, x)$ be the arbitrage free price of a European call with maturity T and strike K when the spot price is x. Recall

$$c(t,x) = xN(d_+) - Ke^{-r\tau} N(d_-)\,, \quad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}} \Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau \Big)\,, \quad \tau = T - t\,.$$

Definition 6.15. The *delta* is $\partial_x c$.

Remark 6.16 (Delta hedging rule). $\Delta_t = \partial_x c(t, S_t)$.

Proposition 6.17. $\partial_x c = N(d_+)$

Definition 6.18. The *Gamma* is $\partial_x^2 c$ and is given by $\partial_x^2 c = \dfrac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\dfrac{-d_+^2}{2}\Big)$.

Definition 6.19. The *Theta* is $\partial_t c$, and is given by $\partial_t c = -rKe^{-r\tau}N(d_-) - \dfrac{\sigma x}{2\sqrt{\tau}}N'(d_+)$

Proposition 6.20. *(1) c is increasing as a function of x.*
(2) c is convex as a function of x.
(3) c is decreasing as a function of t.

Remark 6.21. To properly hedge a short call, you always borrow from the bank. Moreover $\Delta_T = 1$ if $S_T > K$, $\Delta_T = 0$ if $S_T < K$.

Remark 6.22 (Delta neutral, Long Gamma). Say x_0 is the spot price at time t.

- Short $\partial_x c(t, x_0)$ shares, and buy one call option valued at $c(t, x_0)$.
- Put $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$ in the bank.
- What is the portfolio value when if the stock price is x (and we hold our position)?
  - ▷ *(Delta neutral)* Portfolio value $= c(t, x) -$ tangent line.
  - ▷ *(Long gamma)* By convexity, portfolio value is always non-negative.

Remark 6.23. The derivation of the Black–Scholes formula above has a few limitations:

(1) It only applies to markets with one stock.
(2) It requires securities to have a payoff of the form $g(S_T)$.
(3) It can't handle random interest rates.
(4) Deriving the formula (6.4) was so tedious that we skipped it.

We will remedy each of these by providing an alternate approach using *Risk Neutral Measures.*

7. Multi-dimensional Itô calculus

- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

Definition 7.1. The *joint quadratic variation* of X, Y, is defined by

$$[X, Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),$$

Remark 7.2. The joint quadratic variation is sometimes written as $d[X, Y]_t = dX_t \, dY_t$.

Lemma 7.3. $[X, Y]_T = \frac{1}{4}([X + Y, X + Y]_T - [X - Y, X - Y]_T)$

Proposition 7.4 (Product rule). $d(XY)_t = X_t \, dY_t + Y_t dX_t + d[X, Y]_t$

Proposition 7.5. *Say X, Y are two semi-martingales.*

- *Write $X = X_0 + B + M$, where B has bounded variation and M is a martingale.*
- *Write $Y = Y_0 + C + N$, where C has bounded variation and N is a martingale.*
- *Then $d[X, Y]_t = d[M, N]_t$.*

Remark 7.6. Recall, all processes are implicitly assumed to be *adapted* and *continuous.*

Corollary 7.7. *If X is a semi-martingale and B has bounded variation then* $[X, B] = 0$.

Remark 7.8 (Two dimensional chain rule). If X is a differentiable function of t, then

$$d(f(t, X_t, Y_t)) = \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(t, X_t, Y_t)\, dX_t + \partial_y f(t, X_t, Y_t)\, dY_t$$

Remark 7.9 (Notation). $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_x f = \frac{\partial f}{\partial x}$, etc.

Theorem 7.10 (Two-dimensional Itô formula)**.**

- *Let X, Y be a two Itô process.*
- *Let $f = f(t, x, y)$ be a function that's defined for $t \in \mathbb{R}$, $x, y \in \mathbb{R}$.*
- *Suppose $f \in C^{1,2}$. That is:*
  - ▷ *f is once differentiable in t*
  - ▷ *f is twice in both x and y.*
  - ▷ *All the above partial derivatives are continuous. Then:*

$$\begin{aligned} d(f(t, X_t, Y_t)) = \partial_t f(t, X_t, Y_t)\, dt + \partial_x f(t, X_t, Y_t)\, dX_t + \partial_y f(t, X_t, Y_t)\, dY_t \\ + \frac{1}{2}\Big(\partial_x^2 f(t, X_t, Y_t)\, d[X, X]_t + \partial_y^2 f(t, X_t, Y_t)\, d[Y, Y]_t \\ + 2\partial_x \partial_y f(t, X_t, Y_t)\, d[X, Y]_t\Big) \end{aligned}$$

Remark 7.11. We will often drop the arguments of f and simply write

$$\begin{aligned} d(f(t, X_t, Y_t)) = \partial_t f\, dt + \partial_x f\, dX_t + \partial_y f\, dY_t \\ + \frac{1}{2}\Big(\partial_x^2 f\, d[X, X]_t + \partial_y^2 f\, d[Y, Y]_t + 2\partial_x \partial_y f\, d[X, Y]_t\Big) \end{aligned}$$

Remember the arguments are present. After differentiating f you should substitute $x = X_t$, $y = Y_t$.

Remark 7.12 (Integral form). The integral form of the above is

$$\begin{aligned} f(T, X_T, Y_T) - f(0, X_0, Y_0) = \int_0^T \partial_t f\, dt + \int_0^T \partial_x f\, dX_t + \int_0^T \partial_y f\, dY_t \\ + \frac{1}{2}\Big(\int_0^T \partial_x^2 f\, d[X, X]_t + \int_0^T \partial_y^2 f\, d[Y, Y]_t + 2\int_0^T \partial_x \partial_y f\, d[X, Y]_t\Big) \end{aligned}$$

Intuition behind Theorem 7.10.

To use the d-dimensional Itô formula, we need to compute joint quadratic variations.

Proposition 7.13. *Let M, N be continuous martingales, with $\boldsymbol{E} M_t^2 < \infty$ and $\boldsymbol{E} N_t^2 < \infty$.*

(1) *$MN - [M, N]$ is also a continuous martingale.*
(2) *Conversely if $MN - B$ is a continuous martingale for some continuous adapted,* bounded variation *process B with $B_0 = 0$, then $B = [M, N]$.*

Proof.

Proposition 7.14. (1) (Symmetry) $[X, Y] = [Y, X]$
(2) (Bi-linearity) *If $\alpha \in \mathbb{R}$, X, Y, Z are semi-martingales,* $[X, Y + \alpha Z] = [X, Y] + \alpha[X, Z]$.

Proof.

Proposition 7.15. *Let M, N be two martingales, σ, τ two adapted processes.*

- *Let* $X_t = \displaystyle\int_0^t \sigma_s \, dM_s$ *and* $Y_t = \displaystyle\int_0^t \tau_s \, dN_s$.
- *Then* $[X, Y]_t = \int_0^t \sigma_s \, \tau_s \, d[M, N]_s$.

Remark 7.16. Alternately, if $dX_t = \sigma_t \, dM_t$ and $dY_t = \tau_t \, dN_t$, then $d[X, Y]_t = \sigma_t \tau_t \, d[M, N]_t$.

Intuition.

Proposition 7.17. *If M, N are continuous martingales, $\boldsymbol{E} M_t^2 < \infty$, $\boldsymbol{E} N_t^2 < \infty$ and M, N are independent, then $[M, N] = 0$.*

Remark 7.18 (Warning). Independence implies $\boldsymbol{E}(M_t N_t) = \boldsymbol{E} M_t \boldsymbol{E} N_t$. But it *does not* imply $\boldsymbol{E}_s(M_t N_t) = \boldsymbol{E}_s M_t \boldsymbol{E}_s N_t$. So you can't use this to show MN is a martingale, and hence conclude $[M, N] = 0$.

Correct proof.

Remark 7.19. $[M, N] = 0$ does not imply M, N are independent. For example:

- Let $M_t = \int_0^t \mathbf{1}_{\{W_s < 0\}} \, dW_s$
- Let $N_t = \int_0^t \mathbf{1}_{\{W_s \geqslant 0\}} \, dW_s$

Vector Notation.

- *d-dimensional vectors:* Write $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.
- *d-dimensional* random *vectors:* $X = (X_1, \dots, X_d)$, where each X_i is a random variable.
- *d-dimensional stochastic processes:* $X_t = (X_t^1, \dots, X_t^d)$, where each X_t^i is a stochastic process.
  - ▷ For scalars (or random variables): X^i denotes the i-th power of X.
  - ▷ For vectors (or random random vectors): X^i denotes the i-th *coordinate* of X.
  - ▷ There is no ambiguity (can't take powers of vectors, or coordinates of scalars)
- Alternate notation used in many books: Use $X(t)$ for the d-dimensional stochastic process, and $X_i(t)$ for the i-th coordinate.
- Sometimes write $X = (X^1, \dots, X^d)$ for random vectors, instead of $(X_1, \dots, X_d)$.

Theorem 7.20 (Multi-dimensional Itô formula)**.**

- *Let X be a d-dimensional Itô process. $X_t = (X_t^1, \dots, X_t^d)$.*
- *Let $f = f(t, x)$ be a function that's defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.*
- *Suppose $f \in C^{1,2}$. That is:*
  - ▷ *f is once differentiable in t*
  - ▷ *f is twice in each coordinate x_i*
  - ▷ *All the above partial derivatives are continuous. Then:*

$$d(f(t, X_t)) = \partial_t f(t, X_t) \, dt + \sum_{i=1}^d \partial_i f(t, X_t) \, dX_t^i + \frac{1}{2} \sum_{i,j} \partial_i \partial_j f(t, X_t) \, d[X^i, X^j]_t$$

Remark 7.21 (Integral form of Itô's formula).

$$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t) \, dt + \sum_{i=1}^d \int_0^T \partial_i f(t, X_t) \, dX_t^i$$

$$+\frac{1}{2}\sum_{i,j}\int_0^T \partial_i\partial_j f(t,X_t)\,d[X^i,X^j]_t$$

Definition 7.22 (d-dimensional Brownian motion)**.** We say a d-dimensional process $W = (W^1,\dots,W^d)$ is a Brownian motion if:

(1) Each coordinate W^i is a standard 1-dimensional Brownian motion.
(2) For $i \neq j$, the processes W^i and W^j are independent.

Remark 7.23. If W is a d-dimensional Brownian motion then

$$d[W^i,W^j]_t = \begin{cases} dt & i=j\,, \\ 0\,dt & i\neq j\,. \end{cases}$$

Example 7.24. Let $f \in C^{1,2}$, W be a d-dimensional Brownian motion, and set $X_t = f(t,W_t)$. Find the Itô decomposition of X.

Question 7.25. *Let W be a 2-dimensional Brownian motion. Let $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. Is X a martingale?*

Theorem 7.26 (Lévy)**.** *Let M be a d-dimensional process such that:*

(1) M is a continuous martingale.
(2) The joint quadratic variation satisfies: $d[W^i,W^j]_t = \begin{cases} dt & i=j\,, \\ 0\,dt & i\neq j\,. \end{cases}$

Then M is a d-dimensional Brownian motion.

Proof.

8. Risk Neutral Measures

8.1. Risk Neutral Pricing.

Goal.

- Consider a market with a bank and a few stocks. Let S_t^1, S_t^2, ..., S_t^n denote the prices of each stock at time t.
- The bank has interest rate R_t, which is some adapted process.
- Find the risk neutral measure and use it to price securities.

Cash flow.

- Evolution of cash is governed by $\partial_t C_t = R_t C_t$.
- Solving implies $C_t = C_0 \exp\Bigl(\int_0^t R_s\,ds\Bigr)$.

Definition 8.1. Let $D_t = \exp\Bigl(-\int_0^t R_s\,ds\Bigr)$ be the discount factor.

Remark 8.2. Note $\partial_t D = -R_t D_t$.

Remark 8.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

Definition 8.4. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) The discounted price of all stocks is a martingale under $\tilde{P}$. That is, if S^i denotes the price of the i-th stock, then $D_t S_t^i$ is a $\tilde{\boldsymbol{P}}$ martingale.

Theorem 8.5. *The discounted wealth of any self-financing portfolio is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 8.6. The converse requires a "completeness" assumption. If the stocks are modelled by Geometric Brownian motion with a non-degeneracy condition, then we will use the *martingale representation theorem* to show that any martingale under $\tilde{\boldsymbol{P}}$ is the discounted wealth of a self financing portfolio.

Theorem 8.7. *Consider a security that pays V_T at time T. If the security can be replicated, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s\,ds\Big)\,V_T)\Big)\,.$$

Remark 8.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Remark 8.9. We will later study conditions under which any security can be replicated.

8.2. Girsanov Theorem.

Example 8.10. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E}Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \displaystyle\int_A Z_T\,d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}}X = \boldsymbol{E}(Z_T X)$. That is $\displaystyle\int_\Omega X\,d\tilde{\boldsymbol{P}} = \int_\Omega X\,Z_T\,d\boldsymbol{P}$.
- Notation: Write $d\tilde{\boldsymbol{P}} = Z_T\,d\boldsymbol{P}$.

Theorem 8.11 (Cameron, Martin, Girsanov)**.** *Fix $T > 0$, and define:*

- *$b_t = (b_t^1, \dots, b_t^d)$ a d-dimensional adapted process.*
- *W a d-dimensional Brownian motion.*
- *$\tilde{W}_t = W_t + \int_0^t b_s\,ds$ (i.e. $d\tilde{W}_t = b_t\,dt + dW_t$).*
- *$d\tilde{\boldsymbol{P}} = Z_T\,d\boldsymbol{P}$, where*

$$Z_t = \exp\Big(-\int_0^t b_s \cdot dW_s - \frac{1}{2}\int_0^t |b_s|^2\,ds\Big)\,.$$

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

Remark 8.12. Note $\tilde{W}_t$ is a vector.

(1) So $\tilde{W}_t = W_t + \int_0^t b_s\,ds$ means $\tilde{W}_t^i = W_t^i + \int_0^t b_s^i\,ds$, for each $i \in \{1, \dots, d\}$.
(2) Similarly, $d\tilde{W}_t = b_t\,dt + d\tilde{W}_t$ means $d\tilde{W}_t^i = b_t^i\,dt + d\tilde{W}_t^i$ for each $i \in \{1, \dots, d\}$.

Remark 8.13. $\int_0^t b_s \cdot dW_s$ means $\int_0^t \sum_{i=1}^d b_s^i\,dW_s^i$ (dot product).

Proposition 8.14. *$dZ_t = -Z_t b_t \cdot dW_t$. Explicitly, in coordinates,*

$$dZ_t = -Z_t \sum_{i=1}^d b_t^i\,dW_t^i\,.$$

Question 8.15. *Looks like Z is a martingale. Why did we assume it in Theorem 8.11?*

Remark 8.16. We will return and prove Theorem 8.11 later.

8.3. **Constructing Risk Neutral Measures.** Suppose the market has only one stock whose price process satisfies

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t \, .$$

Theorem 8.17. *The (unique) risk neutral measure is given by* $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$*, where*

$$Z_T = \exp\Bigl(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Bigr) \, , \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t} \, .$$

Proposition 8.18. *The stock price satisfies*

$$dS_t = R_t S_t \, dt + \sigma_t S_t \, d\tilde{W} \, ,$$

where $\tilde{W}$ *is a Brownian motion under the risk neutral measure.*

8.4. **Black Scholes Formula revisited.**

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

Theorem 8.19. *Consider a security that pays* $V_T = g(S_T)$ *at maturity time* T*. The arbitrage free price of this security at any time* $t \leqslant T$ *is given by* $f(t, S_t)$*, where*

(6.4)
$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Bigl(x \exp\Bigl(\Bigl(r - \frac{\sigma^2}{2}\Bigr)\tau + \sigma\sqrt{\tau}\, y\Bigr)\Bigr) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}} \, , \qquad \tau = T - t \, .$$

Remark 8.20. This proves Proposition 6.8.

Theorem 8.21 (Black Scholes Formula)**.** *The arbitrage free price of a European call with strike* K *and maturity* T *is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)} N(d_-(T-t,x)) \tag{6.5}$$

where

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}} \Bigl(\ln\Bigl(\frac{x}{K}\Bigr) + \Bigl(r \pm \frac{\sigma^2}{2}\Bigr)\tau\Bigr) \, , \tag{6.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \, , \tag{6.7}$$

is the CDF of a standard normal variable.

Remark 8.22. This proves Corollary 6.9.

8.5. **The Martingale Representation Theorem.**

Theorem 8.23. *If* M_t *is a square integrable martingale with respect to the Brownian filtration, then there exists a* predictable process D *such that* $\boldsymbol{E} \int_0^t D_s^2 \, ds < \infty$ *and*

$$M_t = M_0 + \int_0^t D_s \, dW_s \, .$$

Remark 8.24. A *square integrable* martingale is a martingale for which $\boldsymbol{E} M_t^2 < \infty$ for all t.

Remark 8.25. For our purposes, think of a *predictable process* as a left continuous and adapted process.

Theorem 8.26. *Consider the one stock market form Theorem 8.17.*

(1) *Any $\tilde{\boldsymbol{P}}$ martingale is the discounted wealth of a self financing portfolio (i.e. converse of Theorem 8.5 holds)*
(2) *Any security with an $\mathcal{F}_T$-measurable payoff is replicable, and so Theorem 8.7 holds for any $\mathcal{F}_T$-measurable function V_T.*
(3) *The risk neutral measure is unique.*

8.6. Multi-dimensional market model.

- Let W be a d-dimensional Brownian motion, α a m-dimensional process, and σ a $m \times d$ matrix valued process.
- Let $S^1, \dots, S^m$ be the price processes of m stocks. Set $S = (S^1, \dots, S^m)$.
- Model $dS_t^i = \alpha_i S_t^i \, dt + S_t^i \sum_j \sigma_t^{i,j} \, dW_t^j$.
- Consider a market with the above stocks, and a bank with interest rate given by an adapted process R.

Theorem 8.27. *There is a risk neutral measure* if and only if *you can solve the market price of risk system*

$$\alpha - \vec{R} = \sigma\theta \, .$$

The risk neutral measure is unique if and only if the above system has a unique solution. (Here $\vec{R} = (R, R, \dots, R) \in \mathbb{R}^m$.)

Remark 8.28. Under the risk neutral measure

$$dS_t^i = R S_t^i \, dt + S_t^i \sum_j \sigma_t^{i,j} \, d\tilde{W}_t^j \, .$$

Theorem 8.29 (Fundamental theorems of asset pricing)**.**

(1) *The market has no arbitrage if and only if a risk neutral measure exists.*
(2) *The market is complete and arbitrage free if and only if the risk neutral measure is unique.*

Example 8.30. Consider the above market with $m = 2$, $d = 1$, and α, σ and the interest rate r are all constant in time. The market is complete and arbitrage free if and only if

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2} \, .$$

If the above doesn't hold and explicit arbitrage can be found.

Example 8.31. Consider the above market with $m = 1$, $d = 2$. There are infinitely many risk neutral measures. Can explicitly find securities that can't be replicated. (Or equivalently, can explicitly find processes whose discounted wealth is a $\tilde{\boldsymbol{P}}$ martingale, but are not the wealth of a self financing portfolio.)

8.7. Dividend paying stocks.

- Without dividends, discounted wealth of self-financing portfolios are martingales under the risk neutral measure.
- With dividends, discounted wealth of self-financing portfolios *with the dividends reinvested* are martingales under the risk neutral measure.
- *Model:*
  - ▷ Dividends payed continuously at rate A_t per time unit.
  - ▷ Valid for large composite funds.
  - ▷ $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t - A_t S_t \, dt$.

- *Self financing portfolios:* Wealth evolves according to

$$dX_t = \Delta_t \, dS_t + R_t(X_t - \Delta_t S_t) + \Delta_t A_t S_t \, dt \, .$$

- If X_t is the wealth of a self-financing portfolio which holds one share (and no cash) initially, and then reinvests all dividends then

$$dX_t = \alpha_t X_t \, dt + \sigma_t X_t \, dW_t$$

- *Risk neutral measure:* Chosen to make the discounted wealth of the above self-financing portfolio into a $\tilde{\boldsymbol{P}}$ martingales.

Theorem 8.32. *The risk neutral measure is still given by the formula in Theorem 8.17, and the arbitrage free price of securities are still obtained by Theorem 8.7. Note however, that in this case the stock price satisfies*

$$dS_t = (R_t - A_t) S_t \, dt + \sigma_t S_t \, d\tilde{W} \, , \qquad d(D_t S_t) = -A_t D_t S_t \, dt + \sigma_t D_t S_t \, d\tilde{W} \, ,$$

where $\tilde{W}$ is a Brownian motion under $\tilde{\boldsymbol{P}}$.

Proposition 8.33 (Black Scholes Formula with dividends)**.** *Suppose $A_t = a$, $R_t = r$, and α, σ are constants. The arbitrage free price of a European call with strike K is $c(t, S_t)$ where*

$$c(t,x) = e^{-a\tau} x N(d_+) - K e^{-r\tau} N(d_-) \, , \quad d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}} \Big(\ln\Big(\frac{x}{K}\Big) + \Big(r - a \pm \frac{\sigma^2}{2}\Big)\tau \Big) \, ,$$

8.8. **Forwards and Futures.**

Definition 8.34. A *zero coupon bond* pays \$1 at time T.

Proposition 8.35. *If $\tilde{\boldsymbol{P}}$ is the risk neutral measure, then the arbitrage free price of the bond at time t is*

$$B_{t,T} = \tilde{\boldsymbol{E}}_t \Big(\frac{D_T}{D_t} \Big) \, .$$

Remark 8.36. If $R_t = r$ is constant, then $B_{t,T} = e^{-r(T-t)}$.

Definition 8.37. A *forward contract* is the agreement to buy an asset at price K (called the delivery price) on the delivery date T.

Definition 8.38. The *forward price* at time t is the choice of K for which the forward contract is worth nothing at time t.

Proposition 8.39. *The forward price is given by*

$$\mathrm{For}_t = \frac{S_t}{B_{t,T}} \, .$$

Remark 8.40. Let X_t be the wealth of an investor that buys 1 forward contract at time t_0 (with delivery price $\mathrm{For}_{t_0,T}$). Clearly $X_{t_0} = 0$. However for $t > t_0$, $X_t = S_t - S_{t_0} B_{t,T} / B_{t_0,T}$ which need not be 0. To mitigate risk of default, one can sell the forward contract at time $t_1 > t$, and enter into a new forward contract at time t_1 (with delivery price $\mathrm{For}_{t_1,T}$). One can repeat this again at time $t_2 > t_1$, and so on. *Futures* are designed to do this continuously, without requiring the holder to sell/repurchase contracts.

Definition 8.41. A futures contract delivers the asset (or cash equivalent) to the holder at time T. The holder also pays payments continuously up to maturity time, according to the following:

(1) The futures prices $\mathrm{Fut}_{t,T}$ is chosen so that the contract has 0 value at time t. (The holder pays $\mathrm{Fut}_{t,T}$ to enter into the contract at time t.)
(2) The contract is *marked to margin*: The holder pays $\mathrm{Fut}_{t+dt,T} - \mathrm{Fut}_{t,T}$ over each infinitesimal time interval $[t, t + dt]$. (Note $\mathrm{Fut}_{t+dt,T} - \mathrm{Fut}_{t,T}$ could be positive or negative.)

Proposition 8.42. *The futures price* $\mathrm{Fut}_{t,T}$ *is a* $\tilde{\boldsymbol{P}}$ *martingale, and* $\mathrm{Fut}_{T,T} = S_T$. *Consequently* $\mathrm{Fut}_{t,T} = \tilde{\boldsymbol{E}}_t S_T$.

Remark 8.43. If the interest rate is not random, then $\mathrm{For}_{t,T} = \mathrm{Fut}_{t,T}$. But this need not be true in general.

8.9. **Proof of the Girsanov Theorem.**

Lemma 8.44. *Let* $Z_t = \boldsymbol{E}_t Z_T$. *If* X_t *is* $\mathcal{F}_t$*-measurable, then* $\tilde{\boldsymbol{E}}_s X = \frac{1}{Z_s} \boldsymbol{E}_s(Z_t X_t)$.

Lemma 8.45. *M is a martingale under* $\tilde{\boldsymbol{P}}$ *if and only if ZM is a martingale under* $\boldsymbol{P}$.

Proof of Theorem 8.11.