Continuous Time Finance: Midterm 2.

2024-04-03

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ is the Brownian filtration. Here are a few formulae that you can use:

- Solution formula to the Black Scholes PDE:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,.$$

- Black Scholes Formula for European calls, and the Greeks

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-) \qquad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy\,,$$

$$\partial_x c = N(d_+)\,, \qquad \partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,, \qquad \partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{\sqrt{8\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,.$$

$\boxed{10}$ 1. Let $W = (W^1, W^2)$ be a standard two dimensional Brownian motion, and $Z_t = W_t^1 (W_t^2)^2$. Find $[Z,Z]_t$. Express your answer as $[Z,Z]_T = \int_0^T f(t, W_t)\, dt + \sum_{i=1}^2 \int_0^T g_i(t, W_t)\, dW_t^i$ for some (non-random) functions f, g_1, g_2 that you compute explicitly.

$\boxed{10}$ 2. Consider a market with a bank and a stock. The interest rate is r and the stock price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. A digital share option with strike $K \geqslant 0$ and maturity time T gives the holder one share at maturity, *provided* the stock price is larger than the strike price. The option pays nothing otherwise. Find the arbitrage free price of this option at time $t \leqslant T$. Express your answer in the form $V_t = f(t, S_t)$, and find an explicit formula for the function f without using expectations or integrals. (Your formula may involve the CDF of the standard normal; also note the option pays *one share* at maturity, which is different from the problem on your homework.)

$\boxed{10}$ 3. Suppose S, Z are two stochastic processes such that

$$dS_t = \alpha_t S_t\, dt + \sigma_t S_t dW_t\,, \qquad dZ_t = b_t Z_t\, dW_t\,.$$

Here α_t, σ_t, b_t are adapted processes with $\sigma_t > 0$. If the process $S_t Z_t$ is a martingale, then find b_t in terms of α_t and σ_t. (You may assume $S, Z > 0$ and any required finiteness condition to make Itô integrals martingales.)

$\boxed{10}$ 4. Consider a market with a bank (with interest rate r) and a stock. Let S be a geometric Brownian motion with mean return rate α and volatility σ, modelling the price of a stock. Let $Y_t = \int_0^t S_s\, ds$. Let $g = g(x,y)$ be a function of two variables, and consider a security that pays pays $g(S_T, Y_T)$ at maturity time T. Write down a PDE and terminal conditions such that if a function $f = f(t,x,y)$ is a solution of this PDE, then the security can be replicated, and the wealth of the replicating portfolio at time t is exactly $f(t, S_t, Y_t)$.