Continuous Time Finance: Midterm 1.

2024-02-21

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \,|\, t \geqslant 0\}$ is the Brownian filtration.

10 1. Let $t, \varepsilon > 0$. What is $\lim_{h \to 0^+} \boldsymbol{P}\Big(\dfrac{|W_{t+h} - W_t|}{h^{1/4}} > \varepsilon\Big)$? Justify your answer. (Stating the answer without an explanation will get no credit. If the limit doesn't exist, then say so and justify it appropriately.)

10 2. Let $\mu_t = \boldsymbol{E} X_t$, where X is a stochastic process such that

$$dX_t = -3X_t\, dt + W_t X_t\, dW_t\,.$$

Find a formula for μ_t in terms of μ_0 and t. (Your formula should not involve expectations or integrals.)

10 3. Fix $T > 0$. Decide whether or not the limit

$$\lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^4\,,$$

exists, and justify your answer. Here $t_0 = 0 < t_1 < \cdots < t_n = T$, $P = \{t_0, \ldots, t_n\}$ is a partition of $[0, T]$, and $\|P\| = \max_i (t_{i+1} - t_i)$ is the mesh size of the partition.

10 4. Let $X_t = \displaystyle\int_0^t e^{-r} W_r\, dW_r$. Let $0 < s < t$. Compute $\boldsymbol{E}_s(X_t^2)$. Your final answer may involve unsimplified integrals, as long as the integrand does not involve X_r or W_r for any $r > s$. (Involving X_r, W_r for $r \leqslant s$ is OK.)