Continuous Time Finance: Final.

2024-05-02

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 7 questions and 80 points.*
- *Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ is the Brownian filtration. Here are a few formulae that you can use:

- Solution formula to the Black Scholes PDE:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,.$$

- Black Scholes Formula for European calls, and the Greeks

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-) \qquad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy\,,$$

$$\partial_x c = N(d_+)\,, \qquad \partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,, \qquad \partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{\sqrt{8\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,.$$

- Girsanov's theorem: Let $d\tilde{W}^i = b_t^i\, dt + dW^i$, $d\tilde{\boldsymbol{P}} = Z_T\, dP$ where

$$Z_t = \exp\Big(-\sum_{j=1}^{d} \int_0^t b_s^j\, dW_s^j - \frac{1}{2}\int_0^t |b|_s^2\, ds\Big)$$

If Z is a martingale up to time T, then $\tilde{W}$ is a Brownian motion under $\tilde{\boldsymbol{P}}$ up to time T.

[10] 1. Let X, Z be two independent random variables with $X \sim \mathcal{N}(0,1)$, and $\boldsymbol{P}(Z = 1) = \boldsymbol{P}(Z = -1) = 1/2$. Let $Y = XZ$. Compute the characteristic function of Y.

2. Let W be a 1-dimensional Brownian motion, $t > 1$, $X = 2W_t - W_{t+1} - W_{t-1}$, $Y = W_{t+1} - W_{t-1}$.

[10] (a) Compute $\boldsymbol{E}_t(XY)$ and $\boldsymbol{E}_t X \boldsymbol{E}_t Y$.

[10] (b) Are X and Y uncorrelated? Are they independent? Justify

[10] 3. Let W be a 1 dimensional Brownian motion, and suppose $dS_t = 2tS_t\, dt + (1 + W_t^2)S_t\, dW_t$. Given $0 \leqslant t \leqslant T$, compute $\boldsymbol{E}_t S_T$. Express your final answer without involving expectations, conditional expectations, or integrals.

[10] 4. Consider a market with a bank and one stock. The bank has interest rate $r \in \mathbb{R}$ and the stock price is modelled by a geometric Brownian motion with mean return rate $\alpha \in \mathbb{R}$ and volatility $\sigma > 0$. (The stock does not pay dividends.) Let V_t be the arbitrage free price at time $t \leqslant T$ of a European call option with strike K and maturity T. Let V_t' be the arbitrage free price at time $t \leqslant T$ of a European call option with strike K' and the same maturity time T. Suppose $K < K'$. Decide which of the following statements is necessarily be true, and prove it.

A. $V_t \leqslant V_t'$ almost surely, for every $t \in [0, T)$.

B. $V_t \geqslant V_t'$ almost surely, for every $t \in [0, T)$.

C. For some $t_1, t_2 \in [0, T)$ we have $\boldsymbol{P}(V_{t_1} > V_{t_1}') > 0$ and $\boldsymbol{P}(V_{t_2} < V_{t_2}') > 0$.

[10] 5. Let $b_t = (b_t^1, \dots, b_t^d)$ be a d-dimensional adapted process, W be a d-dimensional Brownian motion, and define

$$Z_t = \exp\Big(-\sum_{j=1}^{d} \int_0^t b_s^j\, dW_s^j - \frac{1}{2}\int_0^t |b|_s^2\, ds\Big)\,.$$

Compute dZ_t and $d[Z,Z]_t$. Express your answers as $\alpha_t\, dt + \sum_{j=1}^{d} \sigma_t^j\, dW_t^j$ for adapted processes $\alpha, \sigma^1, \dots, \sigma^d$ that you find explicitly.

10 6. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants, and the stock does not pay dividends. Let $T > 0$ and consider a derivative security that pays

$$V_T \stackrel{\text{def}}{=} \Big(\ln\Big(\frac{S_T}{S_0}\Big)\Big)^+ = \max\Big\{0, \ln\Big(\frac{S_T}{S_0}\Big)\Big\},$$

at maturity T. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Your final answer should not involve W, $\tilde{W}$, expectations or conditional expectations.

10 7. Consider a market with a bank and m dividend paying stocks. Let R_t denote the interest rate at time t, and S_t^i denote the price of the i^{th} stock at time t. We model the price of the stocks by

$$dS_t^i = \alpha_t^i S_t^i \, dt + S_t^i \sum_{j=1}^{d} \sigma_t^{i,j} \, dW_t^j - A_t^i S_t^i \, dt$$

where $\alpha^i, \sigma^{i,j}, A^i$ are adapted processes and W is a d-dimensional Brownian motion. The process A^i above is the rate at which the i^{th} stock pays dividends. Show how you can find a risk neutral measure in this market by solving the *market price of risk* system of equations. Moreover, find a process $\tilde{W}$ which is a d-dimensional Brownian motion under the risk neutral measure, and express dS^i in terms of $d\tilde{W}$, without using dW.

HINT: We did this in class for stocks that *do not* pay dividends. The answer may be different for dividend paying stocks.