# THE GEOMETRY OF FAIR DIVISION

FLORIAN FRICK

Abstract. We will fairly divide necklaces with various types of beads. We use this problem as a motivating example to explain how geometry naturally appears in solutions for non-geometric problems. The strategy we develop to solve this problem has been used in several other contexts.

## 1. Introduction

Famously, Wigner called the effectiveness of mathematics in the natural sciences "unreasonable." This *Snapshot* explores the entirely reasonable effectiveness of geometry in understanding non-local information. Geometry, as the study of shape, measures phenomena that are global instead of local. If the solution to a mathematical problem depends on the aggregate of the data that is given to us, instead of just a myopic view, one may expect that geometry will be useful in the problem's resolution—even if the problem itself is not a geometric one.

There is a, by now well-established, approach to find solutions for such problems, which has the following outline:

1. Parametrize the space of all potential solutions. (This is a geometric object!)
2. Define a function on this space that measures to which extent a potential solution differs from an actual one.
3. Exploit symmetry to establish that this function has a zero, and thus an actual solution exists.

This proof scheme is of central importance in the field of *Geometric and Topological Combinatorics* and has been used for a multitude of problems ranging from applications in economics (such as the existence of Nash equilibria [7]) to cutting a sandwich with three ingredients with one straight cut such that both halves have the same amount of each ingredient; see Matoušek's book [5] for an excellent introduction. That several problems follow the theme of fair division is not a coincidence; after all, a solution is fair if it is symmetric in some sense. Here we focus on a simple "toy problem" to showcase how modern mathematics applies geometry to numerous (often non-geometric) problems.

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In Section 2 we introduce the problem of fairly dividing a necklace with white and black beads between two people. In Section 3 we add a third type of bead, which complicates the problem. We thus start following the proof scheme above, and recognize the space of potential solutions as a parallelogram. Section 4 defines the function that measures how close we are to a fair division; Section 5 exploits symmetries to show that a solution must always exist. In Section 6 we collect a few examples of other problems, where this proof scheme has been successfully applied.

## 2. Fairly splitting a necklace

Alice and Bob inherited a necklace that consists of two different kinds of beads: Black stones and white stones. The types of stones are not arranged in any particular pattern, and Alice and Bob are unsure about which type of stone is more valuable. Nevertheless, they want to fairly divide the necklace. The only way to ensure that the division is indeed fair is if both Alice and Bob receive the same number of white beads and the same number of black beads. Of course, they would like to disturb the integrity of the necklace as little as possible, that is, they want to use as few cuts as needed to achieve this fair division.

Figure 1. A necklace with two kinds of beads.

Perhaps the necklace to be divided looks like the one depicted above with twelve black stones and twelve white stones. One cut will not suffice to achieve a fair division: This cut would have to be precisely in the middle of the necklace, but then there are seven black stones on the left and only five black stones on the right—unfair! Are two cuts sufficient?

Let's first observe that with two cuts, we separate a piece from the middle of the necklace (and hand it to Alice, say), while Bob gets the outer two pieces. Since Alice is supposed to receive twelve stones in total, the two cuts must be at distance twelve stones. This means that the first cut uniquely determines the second: If the first cut is after the stone in position $j$, the second cut must occur after position $j + 12$. In general, if the necklace has $2n$ stones, the two cuts must be at distance $n$. A second simple observation is that if Alice receives the correct amount of stones in total (twelve in our running example) and she receives the fair amount of black stones (that is, six), she automatically has the right number of white stones too.

These two observations give us a way of seeing that indeed two cuts suffice in general: Call $n$ consecutive stones along the necklace a *window*. Now slide this window, starting with the first $n$ stones, stone-by-stone, all the way to the last $n$ stones. If there was an excess of black stones among the first $n$ stones, then there must be a deficit among the others, the last $n$ stones. Since excess flips to deficit or vice versa, somewhere along the way the window neither contained an excess nor a deficit of black stones; it has the right number of black stones and thus the right number of white stones too. Thus two cuts are always sufficient.

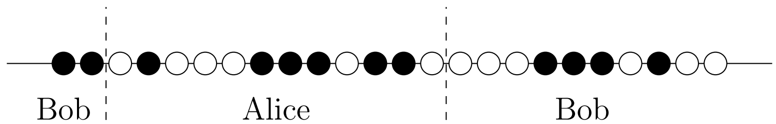


FIGURE 2. A fair division with two cuts.

Sliding the two cuts bounding Alice's piece along the necklace was sufficient to establish the existence of a fair division for two types of beads. Next we will investigate this problem for three types of beads. Here exhibiting the hidden geometry will be crucial. Existence of fair divisions for necklaces with several types of beads were shown by Goldberg and West [3] with simplified proofs by Alon and West [1]. Here we present an elementary proof that is similar in spirit to [3] and [1].

## 3. A parallelogram of possibilities

This fair division problem becomes more intriguing once we allow a third kind of bead. Our sliding window trick already fails for a boring necklace such as the one depicted below. At least three cuts are necessary such that Alice and Bob can receive equal numbers of black, gray, and white stones, respectively.

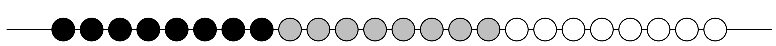

FIGURE 3. A necklace with three kinds of beads.

In order to show that in general three cuts suffice to achieve a fair division, we first describe the general situation: We are given a necklace with $2b$ black stones, $2g$ gray stones, and $2w$ white stones. Our goal is to cut it into four connected pieces $P_1, P_2, P_3, P_4$, such that $P_1$ and $P_3$ together contain $b$ black stones, $g$ gray stones, and $w$ white stones. (And thus the same is true for parts $P_2$ and $P_4$.) We then hand pieces $P_1$ and $P_3$ to Alice and pieces $P_2$ and $P_4$ to Bob.

As in the case of only black and white beads, we first need to parametrize all possible cuts of the necklace. For two types of beads, these possible cuts appeared in a linear order, and we could sweep through the necklace from the first possible division to the last. Now the possible divisions make up a two-dimensional object.

Let $n = b+g+w$. We can think of the necklace as having a stone at every integer between 1 and $2n$. In the plane consider the parallelogram $Q$ with vertices at $(0,0), (0,n), (n,2n)$, and $(n,n)$.

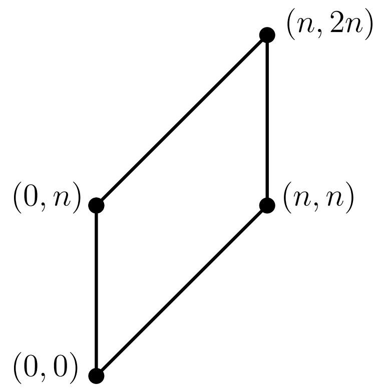


FIGURE 4. The parallelogram $Q$ parametrizing certain divisions of the necklace.

To any point $(x, y)$ with integer coordinates within $Q$, we associate a division of the necklace into four connected pieces $P_1, P_2, P_3, P_4$, such that the total number of stones in pieces $P_1$ and $P_3$ is equal to the total number of stones in pieces $P_2$ and $P_4$. Given such a point $(x, y)$, place the first cut after the bead in position $x$ and the second cut after the bead in position $y$. (Here cutting after the zeroth bead simply means to cut before the first bead.) By definition of $Q$, $x \leq y$, so that the cuts actually appear in this order along the necklace. We allow equality $x = y$, which wastes a cut by performing it twice.

The third cut is now determined by the first two and the requirement that odd-numbered pieces have the same total length as even-numbered pieces. The third cuts needs to occur after the bead in position $z = y - x + n$. These three cuts split the necklace into four pieces. Their lengths are $x, y - x, z - y = n - x$, and $2n - z = x - y + n$. Indeed, the first and the third length sum to $n$, as do the second and the fourth length. Conversely, any such division of the necklace into four pieces corresponds to an integer point within the parallelogram $Q$. Non-integer points in $Q$ also correspond to divisions of the necklace. These divisions, however, may cut through stones and not only between them. Note that a division corresponding to a point on the boundary of $Q$ cuts the necklace into two or three pieces instead of four.

To summarize, the point $(x, y)$ in $Q$ corresponds to the division of the necklace, where we cut in three not necessarily distinct points: At position $x$, at position $y$, and at position $y - x + n$. Think of the necklace as the interval $[0, 2n]$ with a stone at every positive integer. This divides the necklace into pieces $P_1 = [0, x]$, $P_2 = [x, y]$, $P_3 = [y, y - x + n]$, and $P_4 = [y - x + n, 2n]$.

## 4. TESTING FAIRNESS

Now that we have parametrized all valid divisions, let us associate a tuple of two numbers to every point in $Q$: The excess of black beads and the excess of gray beads in the union

$P_1 \cup P_3$ of the first and third piece. That is, to point $(x, y)$ in $Q$ associate the tuple $(\beta - b, \gamma - g)$, where $\beta$ is the number of black beads up to position $x$ in the necklace plus the number of black beads between positions $y$ and $z = y - x + n$. The number $\gamma$ is defined in the same way for gray beads. Denote by $f(x, y)$ the tuple associated to point $(x, y)$.

Suppose that for some point $(x, y)$ in $Q$ we have $f(x, y) = (0, 0)$. Then this means that the first piece $P_1$ and the third piece $P_3$ of the corresponding division together contain $b$ black beads and $g$ gray beads—the correct number. Since their combined length is $n$ stones and $b + g + w = n$, this also implies that the right number $w$ of white beads is in $P_1 \cup P_3$. Thus such a point corresponds to a fair division. We might be worried that if $x$ and $y$ are not integer points, that the division might cut through beads instead of between them. However, if a stone of some color is cut, then since stones appear an even number of times, another stone of that color must also be cut in the same proportions. We can adjust such a pair of cuts simultaneously to not pass through stones. (We sometimes may have to adjust more than two cuts simultaneously.) Thus $f(x, y)$ measures how fair the division corresponding to point $(x, y)$ is, and our goal is to find a fair division, that is, a zero of the function $f$. To find this zero we will investigate the values of $f$ on the boundary of $Q$.

Let us walk along the boundary of $Q$. The vertex $(0, 0)$ of $Q$ is associated with the division, where we cut twice before the first bead (we can neglect those cuts as they do not separate the necklace), and thus the third cut is after bead $z = n$. Thus the first $n$ stones belong to piece $P_3$, while the remaining $n$ stones belong to $P_4$. As we diagonally walk up the edge to vertex $(n, n)$, we increase the length of piece $P_1$, which occupies all stones up to position $x$. Since $x = y$ along the entire edge, the piece $P_2$ always has length zero. Thus along the edge from $(0, 0)$ to $(n, n)$ the union $P_1 \cup P_3$ always occupies the first half of the necklace. Since $P_1 \cup P_3$ is constant along this edge, so is $f$.

Trace up from $(n, n)$ to $(n, 2n)$ and keep track of the divisions parametrized by the points along this edge. Since $x = n$ along this edge, piece $P_1$ remains unchanged; it always consists of the first $n$ stones. The third cut is at position $z = y - x + n = y$, and so it coincides with the second cut. This means that $P_3$ has length zero along this edge, and again $P_1 \cup P_3$ is constantly equal to the first half of the necklace. We have established that $f$ is constant all the way from $(0, 0)$ via $(n, n)$ to $(n, 2n)$.

The other two edges of $Q$ are more interesting. What is the value of $f$ at the vertex $(0, n)$? Recall that $(0, n)$ corresponds to the division, where we cut at positions $0$, $n$, and $n - 0 + n = 2n$. Thus pieces $P_1$ and $P_4$ have length zero, $P_2$ extends from position 0 to $n$, and $P_3$ covers the rest, positions $n$ to $2n$. Compared to the situation in vertex $(0, 0)$, the roles of $P_1 \cup P_3$ and $P_2 \cup P_4$ have swapped; $P_1 \cup P_3$ covers the second half of the necklace in vertex $(0, n)$, but the first half in vertex $(0, 0)$. Thus, an excess of black (respectively gray) beads for one division flips into a deficit for the other, or more concisely $f(0, n) = -f(0, 0)$.

This symmetry extends from the vertices to the two edges incident to $(0, n)$. Points along the top edge of $Q$ are of the form $(a, n + a)$ with $0 \le a \le n$. Such a point corresponds to the division where $P_1$ is the interval $[0, a]$, piece $P_2$ is $[a, n + a]$, and $P_3$ is $[n + a, 2n]$ while $P_4$ is empty. Points along the left edge of $Q$ are of the form $(0, a)$ for $0 \le a \le n$, and such a point gives the division, where $P_1$ is empty, $P_2$ is $[0, a]$, piece $P_3$ is $[a, n + a]$, and $P_4$ is $[n + a, 2n]$. That is, from point $(n, n + a)$ to $(0, a)$ the roles of $P_1 \cup P_3$ and $P_2 \cup P_4$ flip. This means $f(a, n + a) = -f(0, a)$ for all $a$ between 0 and $n$.

## 5. Following paths to find zeros

The topologically inclined reader might have noticed that we are done: The symmetry of $f$ on the boundary of $Q$ implies that $f$ has odd mapping degree around the origin, and thus must have a zero. Thus there is a fair division. Here we prove this central topological fact in an elementary way.

We will exploit the symmetry of the function $f$ on the boundary of $Q$ to find a zero, or equivalently a fair division of the necklace. If $f(0, 0) = (0, 0)$, and thus using one cut precisely in the middle of the necklace already constitutes a fair division, we are done. If not, we may assume that the first coordinate of $f(0, 0)$ is non-zero. Because if only the second coordinate is non-zero, we simply switch the roles of black and gray beads. Denote the first coordinate of $f$, which counts the excess or deficit of black beads in parts $P_1$ and $P_3$, by $f_1$. Similarly, the second coordinate of $f$, measuring the excess or deficit of gray beads in $P_1$ and $P_3$, will be denoted by $f_2$.

Keep track of $f_1$ as we go up the edge of $Q$ that joins $(0, 0)$ to $(0, n)$. Since $f(0, n) = -f(0, 0)$, the sign of $f_1$ flips. This means that along this edge the sign of $f_1$ has to change an odd number of times. Thus, along the edge from $(0, 0)$ to $(0, n)$, $f_1$ is zero an odd number of times. There is only one minor problem with this reasoning: $f_1$ could be zero infinitely many times by being zero along an entire interval. We remedy this by instead considering $f_1 + \frac{1}{2}$. Since $\frac{1}{2} < 1$ and $f_1(0, 0)$ is a non-zero integer, the sign of $f_1 + \frac{1}{2}$ still flips along the left-hand edge of $Q$. Now, since $f_1$ cannot be equal to a non-integer along an interval, we avoid the problem of having infinitely many zeros of $f_1 + \frac{1}{2}$.

By the same reasoning, now using that $f(0, n) = -f(n, 2n)$, $f_1 + \frac{1}{2}$ has an odd number of zeros along the edge joining $(0, n)$ to $(n, 2n)$. Since $f$ is constant on the other two edges of $Q$, there are no more zeros of $f_1 + \frac{1}{2}$ on the boundary of $Q$. We have established that $f_1 + \frac{1}{2}$ has $2k$ zeros on the boundary of $Q$, where $k$ is odd. Furthermore, because of the symmetry of $f$, these zeros come in pairs: For every zero of $f_1 + \frac{1}{2}$ with positive second coordinate $f_2$, there is such a zero with negative $f_2$. So there are exactly $k$ zeros of $f_1 + \frac{1}{2}$ on the boundary of $Q$, where $f_2$ is positive.

We are now in a position to understand the zeros of $f_1 + \frac{1}{2}$ on all of $Q$. Split $Q$ into triangles such that the vertices of the triangles are the points with integer coordinates in $Q$ and edges that only connect vertices whose coordinates differ by at most one, as in Figure 5. Since $f_1$ achieves integer values at points with integer coordinates, $f_1 + \frac{1}{2}$ is never zero at a vertex of one of the small triangles. A zero of $f_1 + \frac{1}{2}$ on an edge of a small triangle occurs if and only if $f_1$ changes sign from one endpoint of the edge to the other. Going around a small triangle the sign of $f_1$ can only change zero or two times; if the sign of $f_1$ is non-constant along the triangle, then it must be constant along one edge and change along the other two edges, which accounts for exactly two sign changes. Thus if $f_1 + \frac{1}{2}$ has a zero somewhere on a small triangle, this triangle contains a line segments of zeros, joining the two zeros of $f_1 + \frac{1}{2}$ on the boundary of the triangle. This implies that zeros of $f_1 + \frac{1}{2}$ come in non-branching paths that either start and end in the boundary of $Q$ or close up to circles.

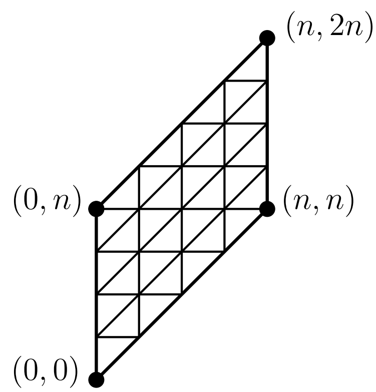


FIGURE 5. Splitting $Q$ into triangles. Here $n = 4$.

We have established that the $2k$ zeros of $f_1 + \frac{1}{2}$ on the boundary of $Q$ are joined in pairs by paths of zeros. Since $k$ is odd, the $k$ zeros of $f_1 + \frac{1}{2}$ with positive $f_2$ cannot be joined in pairs. So at least one of these paths must connect a point on the boundary of $Q$ with positive $f_2$ to a point with negative $f_2$. Since the sign of $f_2$ flips along this path, there must be a point, where $f_2 = 0$. Since the path consists of points with $f_1 + \frac{1}{2} = 0$, we have found a point with $f_1 + \frac{1}{2} = 0$ and $f_2 = 0$. The corresponding division of the necklace fairly divides the gray beads (since $f_2 = 0$) and up to half a bead fairly divides the black beads (since $f_1 + \frac{1}{2} = 0$). Thus the division must cut through a black bead and we can adjust the corresponding cut in such a way that the division of black beads is precisely fair. Then, as we argued before, the white beads are also fairly divided. Finally, we have found a fair division of the necklace using only three cuts.

## 6. Further problems

The reader may be worried that since it took some effort to establish fair divisions for necklaces with three kinds of beads, the reasoning will be much more involved for four or even five kinds of beads. However, the good news is that we have developed all ideas required to fairly divide necklaces with any number of types of beads. For a necklace that consists of beads with $k$ different colors, the space of divisions with $k$ cuts will be a $(k-1)$-dimensional geometric object. Define the function $f$ as before, now measuring the fairness of the division with respect to the first $k-1$ kinds of beads. To establish that $f$ has a zero, either appeal to a topological fact, or give an elementary proof following paths of zeros of the first $k-2$ coordinates of $f$, where the first coordinate was shifted by $\frac{1}{2}$.

The ideas we have presented here can be used in many other contexts and for a variety of problems. Some examples where this proof scheme has been applied include:

- The square peg problem: Does any simple closed curve in the plane have four points that are the vertices of a square? This is still unknown in general, but has been settled with various regularity conditions on the curve; see [6].
- Lovász [4] used this proof scheme to show the following result about intersections of finite sets: If one wants to partition the collection of $k$-element subsets of $\{1, 2, \dots, n\}$ into $c$ parts, such that in each part all sets intersect pairwise, then $c \geq n - 2k + 2$ parts are needed.
- The rent of a 3-bedroom apartment can be split among the three rooms in such a way that the rooms may be assigned to three roommates with their own subjective preferences so that no roommate is envious of another. The same holds true for $n$ roommates in an $n$-bedroom apartment [8], even if the preferences of one roommate are unknown [2].

The reader is invited to identify why these problems benefit from detecting global phenomena and how they depend on inherent symmetries. Finding the relevant space of potential solutions can be easy (for the square peg problem, one may argue with the space of four points on a curve) or the main difficulty: Lovász used the space of probability measures supported on $k$-element sets disjoint from a common $k$-element set to prove his result. New applications of this powerful proof scheme are found regularly.

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