MA 225 Cardinality Worksheet

1. Find the cardinality of each of the sets:

(a) $\{\sin \frac{\pi x}{100} : x \in \mathbb{N}\}$
Finite: 101 elements

(b) $\{\sin x : x \in \mathbb{N}\}$
Countable: show $f(x) = \sin(x)$ is 1-1 and onto.

(c) $\{\sin \frac{\pi x}{100} : x \in \mathbb{R}\}$
Uncountable: this function can be mapped to the interval [-1,1]

(d) $\{(x, y) \in \mathbb{R} \times \mathbb{N} : x + y = \pi \text{ and } x \geq 0\}$
Finite: 3 elements

(e) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y \leq 9\}$
Countable: infinite subset of $\mathbb{Z} \times \mathbb{Z}$ which is countable so this set must also be countable

(f) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y^2 \leq 9\}$
Countable: 29 elements

2. Prove that $\{1, 2\} \times (0, 1)$ is uncountable.
Consider $f : \{1, 2\} \times (0, 1) \to [0, 1]$ is 1-1 and onto. Define

$$f = \begin{cases} \frac{1}{2} + \frac{x}{2} & x = 1 \\ \frac{1}{2} - \frac{x}{2} & x = 2 \\ x & \text{otherwise} \end{cases} .$$

3. Prove that if sets A and B are countable and disjoint, then $A \cup B$ is countable.
Pf: Suppose $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ are 1-1 and onto functions. Define $h : \mathbb{N} \to A \cup B$ by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if n is odd} \\ g\left(\frac{n}{2}\right) & \text{if n is even} \end{cases}$$

Show this function is 1-1 and onto.