MA 355 Homework 9 solutions

1 Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.
Step I: Notice that $f(x)$ is uniformly continuous on $[0, 2]$ because $[0, 2]$ is compact. So given, $\varepsilon > 0, \exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for $|x - y| < \delta$.
Step II: Notice that $f(x)$ is uniformly continous on $[1, \infty)$. For $|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \left| \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{x}} \right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$. Since $x, y \in [1, \infty)$, we see that $\sqrt{x} + \sqrt{y} \geq 2$. So $|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < \frac{|x-y|}{2}$. Thus for $\varepsilon > 0$, take $\delta = 2\varepsilon$. The for $|x - y| < 2\varepsilon$, $|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|} < \frac{|x-y|}{2} < \frac{2\varepsilon}{2} = \varepsilon$.
Step III: Thus for $\varepsilon > 0$, let $\delta = \min(1, 2\varepsilon, \delta_1)$. The for $|x - y| < \delta$, we see that a) x and y are either both in $[0, 2]$ or both in $[1, \infty)$. b) If x, y are both in $[0, 2]$, then $|x - y| < \delta < \delta_1$ implies $|f(x) - f(y)| < \varepsilon$ by Step I. c) If x, y are both in $[1, \infty)$ then $|x - y| < \delta < \frac{\varepsilon}{2}$ implies $|f(x) - f(y)| < \varepsilon$ by Step II. Therefore uniformly continuous.

2 Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ be uniformly continuous on D and suppose $\{x_n\}$ is a Cauchy sequence in D. Then $\{f(x_n)\}$ is a Cauchy sequence.
Pf: Given any $\varepsilon > 0$, since f is uniformly continuous on D there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in D$. Since $\{x_n\}$ is a Cauchy sequence, there exists a number N such that $|x_n - x_m| < \delta$ whenever $m, n > N$. Thus for $m, n > N$ we have $|f(x_n) - f(x_m)| < \varepsilon$, so $\{f(x_n)\}$ is a Cauchy sequence.

3 Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ be uniformly continuous on the bounded set D. Prove that f is bounded on D.
Pf: Suppose $f(D)$ is not bounded.
Claim: There is a sequence $s_n \in D$ such that $f(s_n) \geq n, \forall n$. Pf: Construct the sequence by given $n \in \mathbb{N}$. Define the set $f(D, n) = \{x \in D : g(x) \geq n\}$. This nonempty because $f(D)$ is not bounded. So choose s_n to be any point in $f(D, n)$. Then choose s_{n+1} to be any point in $f(D, n+1)$, etc. We get a sequence of points, s_n such that $f(s_n) \geq n, \forall n$.
Now, since D is bounded we know that $\{s_n\}$ has a convergent subsequence (call it $\{s_{nk}\}$). This subsequence is Cauchy (because all convergent sequences are). Then by the previous problem, $f(\{s_{n_k}\})$ is Cauchy too. Thus $f(\{s_{n_k}\})$ is convergent (because all Cauchy sequences are). But this is impossibly because $f\{s_{n_k}\} \geq n_k$.

#4 Use the definition of derivative to find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
$\lim_{h\to\infty} \frac{f(x+h)-f(x)}{h} = \lim_{h\to\infty} \frac{sqrtx+h-\sqrt{x}}{h} = \lim_{h\to\infty} \frac{sqrtx+h-\sqrt{x}}{h} \left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \right) = \lim_{h\to\infty} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to\infty} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

#5 Let $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ for $x \neq 0$ and $f(0) = 0$.
a) Show that f is differentiable in $\mathbb{R}$.
If $x \neq 0$ then $f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - 2\frac{1}{x} \cos\left(\frac{1}{x^2}\right)$. If $x = 0$ then $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h^2}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right) = 0$.
b) Show that f' is not bounded on the interval $[-1, 1]$.
Assume that f' is bounded. Then there exists $M \ni |f'(x)| \leq M \forall x \in [-1, 1]$. Then take x_0 such that $x_0 > M$, $x_0 > 1$ and $\sqrt{x_0} = n\pi$ for some $n \in \mathbb{N}$. Since M is finite, clearly x_0 exists. Look at $\frac{1}{x_0}$. We see that $\frac{1}{x_0} > 1$, and $|f'\left(\frac{1}{x_0}\right| = |\frac{2}{x_0} \sin(n\pi) - x_0 \cos(n\pi)| = x_0 > M$.