MA 355 Homework 8 solutions

#1 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 3x + 5$. Use the definition (of continuity) to prove that f is continuous at 2.
WTS: $\lim_{x\to 2} f(x) = 3 = f(2)$.
Notice: If $\delta < 1$, then $|x-1| = |x-2+1| \le |x-2|+|1| \le 2$. So $|x^2-3x+2| = |x-1||x-2| < 2|x-2|$.
i) So give $\varepsilon > 0$ (and $\varepsilon \le 2$) take $\delta = \frac{\varepsilon}{2}$ (so $\delta < 1$). Then $|f(x)-3| = |x-2||x-1| \le 2|x-2| = 2\frac{\varepsilon}{2} = \varepsilon$ for $|x-2| < \delta$. ii) Give $\varepsilon > 2$, take $\delta = 1$. Then $|f(x) - 3| \le 2 < \varepsilon$ for $|x-2| < \delta$. Thus given $\varepsilon > 0$, take $\delta = \min(\frac{\varepsilon}{2}, 1)$. Then $|f(x) - 3| < \varepsilon$ for $|x - 2| < \delta$.

#2 Prove: Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that there exists an $M > 0$ and a neighborhood U of c such that $|f(x)| \le M$ for all $x \in U \cap D$.
Pf: Since f is continuous, given $\varepsilon > 0$ (take $\varepsilon = 1$), $\exists\delta > 0$ such that $|f(x)-f(c)| < 1$ for $|x-c| < \delta$. Thus $|f(x)| \le 1 + |f(c)|$ for $|x - c| < \delta$. Take $M = 1 + |f(c)|$ and $|x - c| < \delta$ to be U.

#3 Prove: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $k \in \mathbb{R}$. Prove that the set $f^{-1}(\{k\})$ is closed.
Pf: If $(f^{-1}(\{k\}))$ is NOT closed, then there is a limit point x_0 of $(f^{-1}(\{k\}))$, but x_0 is not an element of $(f^{-1}(\{k\}))$. By definition of limit points, there exist a sequence $s_n \to x_0$ with $s_n \in (f^{-1}(\{k\}))$. Since f is continuous, we know that $\lim(f(s_n)) = f(x_0)$. But $\lim f(s_n) = \lim k = k$ because $s_n \in (f^{-1}(\{k\}))$. So $f(x_0) = k$ implies $x_0 \in (f^{-1}(\{k\}))$. So $(f^{-1}(\{k\}))$ contains all its limit points and is closed.

#4 Suppose f is a real function defined on $\mathbb{R}$ which satisfies $\lim_{h\to 0}[f(x + h) - f(x - h)] = 0$ for every $x \in \mathbb{R}$. Does this imply f is continuous?
No. Consider

$$f(x) = \begin{cases} \frac{1}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is not continuous at $x = 0$ even though $f(h) - f(-h) = 0$ for all $x \in \mathbb{R}$.

#5 If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\bar{E}) \subset \overline{f(E)}$ for every set $E \subset X$.
Pf: For every $x \in E, f(x) \in f(E) \subset \overline{f(E)}$, hence $x \in f^{-1}(\overline{f(E)})$. Thus $E \subset f^{-1}(\overline{f(E)})$. The last set must be close as the preimage of the closed set $\overline{f(E)}$ (corollary to 4.8), hence it also contains $\bar{E}$. So, $\bar{E} \subset f^{-1}(\overline{f(E)}) \implies f(\bar{E}) \subset f(f^{-1}(\overline{f(E)})) \subset \overline{f(E)}$.

6 Show the equation $3^x = x^2$ has at least one real solution.
Look at $g(x) = 3^x - x^2$. We see $g(-1) = \frac{-2}{3}$ and $g(0) = 1$. Thus by the Intermediate Value Theorem, there exists a $c \in (-1, 0)$ such that $g(0) = 0$ or $3^c = c^2$.