MA 355 Homework 7 solutions

#1 Use the definition to prove $lim_{x\to 5}x^2-3x+1=11$. Notice if $|x-5|<\delta$ and $\delta\le 1$, then $|x+2|=|x-5+7|\le|x-5|+7\le 8$. So consider two cases: A) Given $\varepsilon>0$ (and $\varepsilon\le 8$), choose $\delta=\frac{\varepsilon}{8}$ (thus $\delta<1$) and we see that $|x^2-3x+1-11|=|x-5||x+2|<8|x-5|<8\frac{\varepsilon}{8}=8$ for $|x-5|<\delta=\frac{\varepsilon}{8}$. B) Given $\varepsilon>0$ (and $\varepsilon>8$), choose $\delta=1$. We see that $|x^2-3x+1-11|=|x-5||x+2|<8<\varepsilon$ for $|x-5|<1$. So given $\varepsilon>0$ choose $\delta=\min(1,\frac{\varepsilon}{8})$.

#2 Let $D\subset\mathbb{R}$. Let f,g,h be functions from D into $\mathbb{R}$ and let c be a limit point of D. Suppose $f(x)\le g(x)\le h(x)$, for all $x\in D$ with $x\ne c$, and suppose that $\lim_{x\to c}f(x)=\lim_{x\to c}h(x)=L$. Prove that $\lim_{x\to c}g(x)=L$.
Pf: Because $\lim_{x\to c}h(x)=L$ we know that given $\varepsilon>0,\exists\delta_1>0$ such that $|h(x)-L|<\varepsilon$ for $|x-c|<\delta_1$. Similarly, since $\lim_{x\to c}f(x)=L$ we know that given $\varepsilon>0,\exists\delta_2>0$ such that $|f(x)-L|<\varepsilon$ for $|x-c|<\delta_2$. We know that $g(x)-L\le h(x)-L\le|h(x)-L|$ and $-(g(x)-L)\le-(f(x)-L)\le|f(x)-L|$. Given ε, the for $\delta=\min\{\delta_1,\delta_2\}$, we have $g(x)-L<|h(x)-L|<\varepsilon$ and $-(g(x)-L)<|f(x)-L|<\varepsilon$ so $|g(x)-L|<\varepsilon$ for $|x-c|<\delta$.

#3 Prove: Let $D\subset\mathbb{R}$. If $f:D\to\mathbb{R}$ and if c is a limit point of D, then f can have only one limit at c.
Pf: Suppose $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}f(x)=M$ where $L\ne M$. Define $\varepsilon=\frac{|L-M|}{3}$. So by definition, if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$ and $|f(x)-M|<\varepsilon$.

$$\begin{aligned}|L-M| &= |L-M+f(x)-f(x)|=|L-f(x)-M+f(x)|\le|-1||f(x)-L|+|f(x)-M|\\ &= |f(x)-L|+|f(x)-M|<2\varepsilon.\end{aligned}$$

So $|L-M|+\varepsilon<3\varepsilon=3\left(\frac{|L-M|}{3}\right)$. But this implies $|L-M|+\varepsilon<|L-M|$ which is clearly a contradiction.

#4 Prove: Let $D\subset\mathbb{R}$. If $\lim_{x\to a}f(x)=L$ then $\lim_{x\to a}|f(x)|=|L|$.
Pf: First we prove the claim: $||x|-|y||<|x-y|$ where $|x|=\sqrt{x^2}$. Square the LHS. Then $||x|-|y||^2=(|x|-|y|)^2=|x|^2-2|x||y|+|y|^2=x^2-2|x||y|+y^2$. Now square RHS, $|x-y|^2=x^2-2xy+y^2$. Then since $2|x||y|=2|xy|\ge 2xy$ we see $||x|-|y||<|x-y|$.
Now if $\lim_{x\to a}f(x)=L$, then given $\varepsilon>0,\exists\delta>0$ such that $|f(x)-L|<\varepsilon$ for $|x-a|<\delta$. But by the claim se wee that $||f(x)|-|L||\le|f(x)-L|<\varepsilon$ for $|x-a|<\delta$ and thus $\lim_{x\to a}|f(x)|=|L|$.