MA 355 Homework 4 solutions

#1 Prove: If S is a nonempty closed, bounded subset of $\mathbb{R}$, then S has a maximum and a minimum.

Pf: Since S is bounded above, $m = supS$ exists by the completeness of $\mathbb{R}$. Since m is the least upper bound for S, given any $\varepsilon > 0$, $m - \varepsilon$ is not an upper bound for S. If $m \ni S$, this implies that there exists $x \in S$ such that $m - \varepsilon < x < m$. But this implies that m is a limit point of S. Since S is closed, it contains all of its limit points. Thus we much have $m \in S$, and we conclude that $m = maxS$. Similarly, $infS \in S$ so $infS = minS$.

#2 Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Prove that K is compact directly from the definition.

Pf: Let $\cup_\alpha U_\alpha$ be an open cover of K. The point 0 belongs to one of the sets U_α, say U_{α_0}. Since U_{α_0} is open, it contains a neighborhood of 0 of some radius r. The points $\frac{1}{n}$ for $n > \frac{1}{r}$ will be therefore contained in U_{α_0} as well. There are only finitely many remaining points $\frac{1}{n}$, $n \leq \frac{1}{r}$; each of them contained in some U_{α_n}. So, the union $U_{\alpha_a} \cup_{n \leq \frac{1}{r}} U_{\alpha_n}$ is a finite subcover for K. Thus, K is compact.

3 Give an example of an open cover of the segment $[0, 1)$ which has no finite subcover.

Consider the union $\cup_{n \in \mathbb{N}, n>1} U_n$ where $U_n = (-1, 1 - \frac{1}{n})$. Every point $x \in [0, 1)$ is in that union, since $x \in U_n$ whenever $N > \frac{1}{1-x}$. On the other hand, if we take only finitely many intervals U_n, then their union coincides with one of them , precisely with the interval indexed by the largest integer n. That set does not contain the interval $(1 - \frac{1}{n}, 1)$, so does not cover $[0, 1)$.

4 Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dens in [0,1]? Is E compact?

The set E is uncountable, since it is in 1-1 correspondence with the set of all binary sequences. The set E is not dense in [0,1]. The set E is compact because it is closed (show contains all limit points) and bounded.

5 Show that the Bolzano-Weierstrass theorem does not necessarily hold in all arbitrary ordered fields F. (Hint: Consider $F = \mathbb{Q}$)

Consider $B = \{1.4, 1.41, 1.414, 1.4142, ...\}$, the set of rational numbers that approximate $\sqrt{2}$. Clearly this set is bounded and infinite. But there is no limit point in $\mathbb{Q}$.