MA 355 Homework 3 solutions

#1Find the interior of the following sets:

$$(i)\ \left\{\frac{1}{n} : n \in \mathbb{N}\right\} : \emptyset$$
$$(ii)\ [0,3] \cup (3,5) : (0,5)$$
$$(iii)\ [0,2] \cap [2,4] : \emptyset$$

#2 Classify the following sets as open, closed or neither:

$$(i)\ \left\{\frac{1}{n} : n \in \mathbb{N}\right\} : neither$$
$$(ii)\ \mathbb{N} : closed$$
$$(iii)\ \{x : x^2 > 0\} : open$$

#3 Find the closure of the above sets

$$(i)\ \left\{\frac{1}{n} : n \in \mathbb{N}\right\} : \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$$
$$(ii)\ \mathbb{N} : \mathbb{N}$$
$$(iii)\ \{x : x^2 > 0\} : \mathbb{R}$$

#4 Show $\emptyset$ is both open and closed.
Open: For a set not to be open, at least one of its points must fail to be an interior point. Therefore the set must not be empty. **Closed**: $\emptyset^C = \mathbb{R}$. But $\mathbb{R}$ is open since $int(\mathbb{R}) = \mathbb{R}$. Thus $\emptyset$ is closed.

#5 Show $cl(S \cap T) \subset cl(s) \cap cl(T)$.
Suppose $x \in cl(S \cap T)$, then $\forall \varepsilon > 0, N_\varepsilon(x) \cap S \cap T \neq \emptyset$, which implies $\forall \varepsilon > 0$, $N_\varepsilon(x) \cap S \neq \emptyset$ and $\forall \varepsilon > 0$, $N_\varepsilon(x) \cap T \neq \emptyset$. Thus $x \in cl(S) \cap cl(T)$. $\therefore cl(S \cap T) \subset cl(s) \cap cl(T)$.

#6 If $B_n = \cup_{i=1}^n A_i$, prove that $\cup_{i=1}^n \bar{A}_i \subset \bar{B}_n$.
Suppose $p \in \cup_{i=1}^n \bar{A}_i$ then $p \in \bar{A}_i$ for some $i = \{1, 2, \dots n\}$. If $p \in A_i$ then $p \in B_n$ since B_n is the union of all the $A_i's$. Now suppose $p \notin A_i$ then $p \in A_i'$ for some $i \in \{1, 2, \dots n\}$. Then for all $\varepsilon > 0$ and $N_\varepsilon(p)$ there exists a $q = N_\varepsilon(p) \cap A_i$. But that means $q \in A_i$ which means $q \in B_n$. Thus p is a limit point of B_n which means $p \in \bar{B}_n$. $\therefore \cup_{i=1}^n \bar{A}_i \subset \bar{B}_n$.

7 Let E^o denote the set of all interior points of a set E. Prove that E is open iff $E^o = E$.
($\Rightarrow$) Suppose E is open. Then every point of E is an interior point of E. Then $E \subset E^o$. It is obvious $E^o \subset E$ because all interior points of E are from E by definition. So $E^o = E$.
($\Leftarrow$) Suppose $E^o = E$ then by definition E^o is open, so E must then be open.

#8 If x is a limit point of a set S, then every neighborhood of x contains infinitely many points. Pf: (by contr.) Suppose there exists a neighborhood of x containing only finitely many points belonging to S. Let $p_1, p_2, ..p_n$ be such points and let $\varepsilon = \min_i d(x, p_i)$. Then the neighborhood of x with radius $\frac{\varepsilon}{2}$ does not have any $y \in S$ such that $x \neq y$. This contradicts x being a limit point.