MA 355 Homework 2 solutions

#1 Let A, B, C be sets and let $f : A \to B$, $g : B \to C$ be functions. Prove: If f is onto B and g is onto C, then $g \circ f : A \to C$ is onto C.
Suppose f is onto B and g is onto C. Let $c \in C$. Then there is a $b \in B$ such that $g(b) = c$, since g is onto B. Also there is an $a \in A$ such that $f(a) = b$ since f is onto A. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$. Thus $g \circ f$ is onto. #

2 Show the relation $\sim$ (two sets are equivalent) is an equivalence relation. • Reflexive: For any set A, let $I_A : A \to A$ be the identity relation, which is clearly bijective. Thus $\sim$ is reflexive.
• Symmetric: For sets A, B, suppose $A \sim B$. Then there is a 1-1 correspondence $f : A \to B$. So f^{-1} exists and $f^{-1} : B \to A$. Therefore, $\sim$ is symmetric.
• Transitive: Let A, B, C be sets such that $A \sim B$ and $B \sim C$. Then there are 1-1 correspondences $f : A \to B$ and $g : B \to C$. So $g \circ f : A \to C$ is also a 1-1correspondence. Thus $\sim$ is transitive.

#3 Give an example of a countable collection of finite sets whose union is not finite.
Define $A_n = n$ where $n \in \mathbb{N}$. Then $\cup_n A_n = \mathbb{N}$ which is infinite.

4 Are the following sets finite, countable or uncountable? Explain or prove your answer in each case.
(i) $\{(x, y) \in \mathbb{N} \times \mathbb{R} : xy = 1\}$ is countable. Let $f(n) = (n, \frac{1}{n})$. Then $f : \mathbb{N} \to A$ is 1-1 and onto.
(ii)$\left(\frac{1}{4}, \frac{3}{4}\right)$ is **uncountable**. Define $f : (0, 1) \to \left(\frac{1}{4}, \frac{3}{4}\right)$ by $f(x) = x - \frac{1}{4}$.

#5 Is the set of all irrational numbers countable? Prove your answer.
The set $\mathbb{R}$ of all real numbers is the (disjoint) union of the sets of all rational and irrational numbers. We know that $\mathbb{R}$ is uncountable, whereas $\mathbb{Q}$ is countable. If the set of all irrational numbers were countable, then $\mathbb{R}$ would be the union of two countable sets, hence countable. Thus the set of all irrational numbers is uncountable.

#6 Let $\mathbb{N}$ be the set of natural numbers. Prove that $\mathbb{N} \times \mathbb{N}$ is countable.
(1-1): Suppose $f(m, n) = f(p, q)$. Then $2^{m-1}(2n - 1) = 2^{p-1}(2q - 1)$. Suppose $m \neq p$ and WLOG $m > p$. Dividing by 2^{p-1}, $2^{m-p}(2n - 1) = 2q - 1$. The left hand side is even and the right is odd. This contradiction shows $m = p$. Then $2n - 1 = 2q - 1$ which implies $n = q$.
(onto): Suppose $z \in \mathbb{N}$. (i) Suppose z is odd. Let $m = 1$, $n = \frac{z+1}{2}$. Then $f(m, n) = 2^0(2\frac{z+1}{2} - 1) = z$.
(ii) Suppose z is even. Let m be the largest integer such that z is divisible by 2^{m-1}. Then $m \geq 2$ and $z = k2^{m-1}$, where k is an odd integer. Let $n = \frac{k+1}{2}$. Then $f(m, n) = 2^{m-1}(2n - 1) = z$. $\therefore f$ is onto $\mathbb{N}$.