# MA 355 Homework 1

• For each subset of $\mathbb{R}$, give its maximum, minimum, supremum, and infimum, if they exist:
$\{1, 3\}$: min and inf=1, max and sup=3
$\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$: min and inf=$\frac{1}{2}$, max=none, sup=1
$(-\infty, 4)$: min and inf=none, max=none, sup=4

p.21: #1. (a) Suppose $r \in \mathbb{Q}$ and $x$ is irrational and $r + x$ is rational. Then $r = \frac{p}{q}$ and $r + x = \frac{s}{t}$ for $p, q, p, s, t \in \mathbb{Z}$ where $q, t \neq 0$. Thus $x = (r + x) - r = \frac{s}{t} - \frac{p}{q} = \frac{sq-pt}{tq}$. Since $sq - pt$ and $tq$ are integers and $tq \neq 0$, $x$ is rational. This is a contradiction. $\therefore$ If $r$ is rational and $x$ is irrational then $r + x$ is irrational.
(b) (a) Suppose $r \in \mathbb{Q}$ and $x$ is irrational and $rx$ is rational. Then $r = \frac{p}{q}$ and $rx = \frac{s}{t}$ for $p, q, p, s, t \in \mathbb{Z}$ where $q, t \neq 0$. Thus $x = rx\frac{1}{r} = \frac{s}{t} * \frac{p}{q} = \frac{sp}{tq}$. Since $sp$ and $tq$ are integers and $tq \neq 0$, $x$ is rational. This is a contradiction. $\therefore$ If $r$ is rational and $x$ is irrational then $rx$ is irrational.

#2. Suppose there exists a rational number $r$ whose square is 12. Write r in lowest terms, $r = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and have no factors in common. Then $m^2 = 12n^2 = 3 \times 2^2 \times n^2$. Since 3 appears in the right-hand side, 3 divides $m^2$. But 3 is prime, so it must divide $m$; hence $m^2$ must be divisible by 9. But then 3 divides $n^2$ and therefore n as well, contrary to the assumption that $m$ and $n$ have no common factors. Thus there is no rational number whose square is 12.

# 4. Since $E$ is nonempty, it has at least one element, say, x. Since $\alpha$ is a lower bound of $E$, we know $\alpha \leq x$. Similary, since $\beta$ is an upper bound of $E$, $x \leq \beta$. By the transitivity of the order $\leq$ we conclude $\alpha \leq \beta$.

# 5. Since $A$ is nonempty and bounded below, there exists $infA$. By definition, $infA \leq x, \forall x \in A$, so $-infA \geq -x, \forall x \in A$, i.e., $-infA \geq y, \forall y \in -A$. Thus $-infA$ is an upper bound of $-A$. The set $-A$ is thus nonempty and bounded above, hence there exists $sup(-A)$. Since $sup(-A) \geq y, \forall y \in -A, -sup(-A) \leq -y, \forall y \in -A, i.e., -sup(-A) \leq x, \forall x \in A$. Thus $-sup(-A)$ is a lower bound of A. Now $-\inf A \geq sup(-A), -sup(-A) \leq infA$ since a lower bound is not bigger than the greatest lower bound and an upper bound is not smaller than the least upper bound. But these last inequalities are equivalent to $-infA \geq sup(-A), -infA \leq sup(-A)$. Hence $-infA = sup(A)$.

• Suppose $x, y \in \mathbb{R}$ and $x < y$. Consider $\tilde{x} = \frac{x}{\sqrt{2}}$ and $\tilde{y} = \frac{y}{\sqrt{2}}$. By the density of the reals, there exists a rational number $r$ such that $\tilde{x} < r < \tilde{y}$ which implies $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Define $w = r\sqrt{2}$ which is irrational by a previous $\#1b$.