MA 355 Homework 10 solutions

#1 Use the mean value theorem to establish:
a) $\frac{1}{8} < \sqrt{51} - 7 < \frac{1}{7}$
Look at $f(x) = \sqrt{49+x}$ on $[0,2]$. Then by MVT there is $c \in (0,2)$ such that $f'(c) = \frac{f(2)-f(0)}{2} = \frac{1}{2\sqrt{49+c}}$ which implies $f(2) - f(0) = \frac{1}{\sqrt{49+c}}$. Thus $\sqrt{51} - 7 = \frac{1}{\sqrt{49+c}}$. Since $7 < \sqrt{49+c} < \sqrt{51} < 8$ we see that $\frac{1}{7} > \frac{1}{\sqrt{49+1}} > \frac{1}{8}$. Thus $\frac{1}{8} < \sqrt{51} - 7 < \frac{1}{7}$.
b) $|\cos(x) - \cos(y)| \leq |x - y|$ for $x, y \in \mathbb{R}$
Consider the interval $[y, x]$ and $g(x) = \cos(x)$. By the MVT, we know there exists $c \in (y, x)$ such that $f(x) - f(y) = f'(c)(x - y)$. Taking absolute values we see $|f(x) - f(y)| = |f'(c)||x - y|$. Substituting gives $|\cos(x) - \cos(y)| = |-sin(c)||x - y|$. But $0 \leq |\sin(x)| \leq 1$. So $|\cos(x) - \cos(y)| \leq |x - y|$.

#2 Suppose i) f is continuous for $x \geq 0$, ii) $f'(x)$ exists for $x > 0$, iii) $f(0) = 0$, iv) f' is monotonically increasing. Define $g(x) = \frac{f(x)}{x}, x > 0$ and prove g is monotonically increasing.
Notice $g'(x) > 0 \forall x > 0 \iff g'(x) = \frac{xf'(x) - f(x)}{x^2} > 0 \iff f'(x) > \frac{f(x)}{x}$. Since $f'(x)$ exists, $f(x) - f(0) = f'(c)(x - 0)$ where $0 < c < x$ by MVT. Thus $f'(c) = \frac{f(x)}{x}$ where $0 < c < x$. Since f' is monotonically increasing, $f'(x) > f'(c)$, thus $f'(x) > \frac{f(x)}{x}$ for all $x > 0$.

#3 Let f be defined on an interval I. Suppose there exists $M > 0$ and $\alpha > 0$ such that $|f(x) - f(y)| \leq M|x - y|^\alpha$ for all $x, y \in I$. (Such a function is said to satisfy a Lipschitz condition of order α on I.)
a) Prove that f is uniformly continuous on I.
Let $\varepsilon > 0$. Define $\delta = \left(\frac{\varepsilon}{M^\alpha}\right)^{\frac{1}{\alpha}}$. Assume $|x - y| < \delta$. Then $|f(x) - f(y)| \leq M|x - y|^\alpha < M\delta^\alpha = \varepsilon$.
b) If $\alpha > 1$, prove that f is constant on I. (Hint: First show that f is differentiable on I.)
Suppose $|f(x) - f(y)| \leq M|x - y|^\alpha$ then $0 \leq \left|\frac{f(x)-f(y)}{x-y}\right| \leq |x - y|^{\alpha - 1}$ for all $x \neq y \in \mathbb{R}$. Note that the left and right sides of the expression tend to 0 as $x \to y$. Thus by the sandwich theorem, $\lim_{x \to y} \left|\frac{f(x)-f(y)}{x-y}\right| = 0 \forall y \in \mathbb{R}$. Hence $f' = 0$ everywhere in $\mathbb{R}$, so f is constant.
c) Show by an example that if $\alpha = 1$, then f is not necessarily differentiable on I.
$f(x) = |x|$
d) Let $\alpha = 1$. Prove that if g is differentiable on an interval I and if g' is bounded on I, then g satisfies a Lipschitz condition of order 1 on I. Suppose g is differentiable. Then $\lim_{x \to c} \frac{g(x)-g(c)}{x-c} = g'(c)$. By the MVT $\frac{|g(x)-g(y)|}{|x-y|} = |g'(k)|, k \in (x, y)$. This implies $|g(x) - g(y)| = |g'(k)||x - y|$. But we know $|g'(x)|$ exists and is bounded, let's say by M. Thus $|g(x) - g(y)| \leq M|x - y|$.
#4 Evaluate the following limits.
a)$\lim_{x \to 1} \frac{\ln x}{x-1}$
Notice $\ln(1) = 0$ and $x - 1 = 0$. So apply L'Hospital's Rule. Then $\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = 1$.
b)$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
Notice $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} e^{\ln\left(1 + \frac{1}{x}\right)^x}$. So let's examine $\lim_{x \to \infty} \ln\left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$. Notice that both the numerator and denominator go to 0, so by L'Hospital we

have $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(\frac{-1}{x}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$. Therefore $\left(1+\frac{1}{x}\right)^x \to e$.

c)$\lim_{x\to 0}\frac{\tan x - x}{x^3}$

Observe the limit goes to $\frac{0}{0}$, So we apply L'Hospital (3 times) to see $\lim_{x\to 0}\frac{\tan x - x}{x^3} = \frac{1}{3}$.

5 If $f(x) = |x|^3$, compute $f'(x), f''(x)$ for all real x, and show that $f^{(3)}(x)$ does not exist.

Since

$$f(x) = |x|^3 = \begin{cases} x^3 & \text{if } x \geq -, \\ -x^3 & \text{if } x < 0 \end{cases} \tag{0.1}$$

we get

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \geq -, \\ -3x^2 & \text{if } x < 0 \end{cases} \tag{0.2}$$

and $f'(0) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = 0$. Also,

$$f''(x) = \begin{cases} 6x & \text{if } x \geq -, \\ -6x & \text{if } x < 0 \end{cases} \tag{0.3}$$

and $f''(0) = \lim_{x\to 0^+} f''(x) = \lim_{x\to 0^-} f''(x) = 0$.

But then $\lim_{x\to 0^+}\frac{f''(x)-f''(0)}{x} = \lim_{x\to 0^+}\frac{6x-0}{x} = 6 \neq -6 = \lim_{x\to 0^-}\frac{-6x-0}{x} = \lim_{x\to 0^-}\frac{f''(x)-f''(0)}{x}$ so $f'''(0)$ does not exist.

6 A function $f : D \to \mathbb{R}$ is said to have a local maximum (minimum) at a point $x_0 \in D$ if there is a neighborhood U of x_0 such that $f(x) \leq f(x_0)$ ($f(x) \geq f(x_0)$) for all $x \in U \cap D$. Suppose for some integer $n \geq 2$ that the derivatives $f', f'', f''', ..f^{(n)}$ exist and are continuous on an open interval I containing x_0 and that $f'(x_0) - \cdots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Use Taylor's Theorem to prove:

a) If n is even then $f^{(n)} < 0$ then f has a local maximum at x_0

similar to b

b) If n is even then $f^{(n)} > 0$ then f has a local minimum at x_0

We know we can express $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + ... + \frac{f^{(n)}(x)}{n!}(x-x_0)^n$. Using that $f^{(k)} = 0$ for all $k = 1, ...n-1$ we see that $f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$ where $c \in (x, x_0)$. We know that $f^{(n)}(x_0) \neq 0$, and $f^{(n)}(x_0)$ is continuous, so there is a nbhd U of x_0 on which $f^{(n)}(x_0) \neq 0$. Thus on U $\frac{f^{(n)}(x)}{n!}(x-x_0)^n$ is positive if $f^{(n)}(x_0)$ is. So if $f^{(n)}(x_0) > 0$, $f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)$ on U, so $f(x_0)$ is a local minimum.

c) If n is odd then f has neither a local maximum nor a local minimum at x_0.

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