MA 355 Homework 10

#1 Use the mean value theorem to establish:
a) $\frac{1}{8} < \sqrt{51} - 7 < \frac{1}{7}$
b) $|\cos(x) - \cos(y)| \leq |x - y|$ for $x, y \in \mathbb{R}$

#2 Suppose i) f is continuous for $x \geq 0$, ii) $f'(x)$ exists for $x > 0$, iii) $f(0) = 0$, iv) f' is monotonically increasing. Define $g(x) = \frac{f(x)}{x}$, $x > 0$ and prove g is monotonically increasing.

#3 Let f be defined on an interval I. Suppose there exists $M > 0$ and $\alpha > 0$ such that $|f(x) - f(y)| \leq M|x - y|^\alpha$ for all $x, y \in I$. (Such a function is said to satisfy a Lipschitz condition of order α on I.)
a) Prove that f is uniformly continuous on I.
b) If $\alpha > 1$, prove that f is constant on I. (Hint: First show that f is differentiable on I.)
c) Show by an example that if $\alpha = 1$, then f is not necessarily differentiable on I.
d) Let $\alpha = 1$. Prove that if g is differentiable on an interval I and if g' is bounded on I, then g satisfies a Lipschitz condition of order 1 on I.

#4 Evaluate the following limits.
a)$\lim_{x \to 1} \frac{\ln x}{x - 1}$
b)$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
c)$\lim_{x \to 0} \frac{\tan x - x}{x^3}$

5 If $f(x) = |x|^3$, compute $f'(x), f''(x)$ for all real x, and show that $f^{(3)}(x)$ does not exist.

6 A function $f : D \to \mathbb{R}$ is said to have a local maximum (minimum) at a point $x_0 \in D$ if there is a neighborhood U of x_0 such that $f(x) \leq f(x_0)$ $(f(x) \geq f(x_0))$ for all $x \in U \cap D$. Suppose for some integer $n \geq 2$ that the derivatives $f', f'', f''', ..f^{(n)}$ exist and are continuous on an open interval I containing x_0 and that $f'(x_0) - \cdots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Use Taylor's Theorem to prove:
a) If n is even and $f^{(n)} < 0$ then f has a local maximum at x_0
b) If n is even and $f^{(n)} > 0$ then f has a local minimum at x_0
c) If n is odd and f has neither a local maximum nor a local minimum at x_0.