21-355 Real Analysis Second Exam Fall, 2011

1. Give a precise mathematical definition or statement of:

- A function $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ and $c \in D$ is continuous.
  f is continuous at c if $\forall \varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ whenever $x \in D$.
- The Extreme Value Theorem
  Let $D \subset \mathbb{R}$ and D is compact and suppose $f : D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D. That is there exist points x_1 and x_2 in D such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in D$.
- The Chain Rule
  Let I, J be intervals in $\mathbb{R}$, let $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ where $F(I) \subset J$, and let $c \in I$. If f is differentiable at c, and g is differentiable at $f(c)$, the $g \circ f$ is differentiable at c, and $(g \circ f)'(c) = g'(f(c))f'(c)$.
- The Mean Value Theorem
  Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a))}{b-a}$.

2.(a) Suppose $f : [a, b] \to [a, b]$ is continuous. Prove that f has a fixed point. That is, prove that there exists a $c \in [a, b]$ such that $f(c) = c$.
Pf: If $f(a) = a$ or $f(b) = b$, then done. So assume $f(a) > a$ and $f(b) < b$. Consider the function $g(x) = f(x) - x$ which is clearly continuous. We see that $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. So, by the IVT we see there exists $c \in (a, b)$ such that $g(c) = 0$. But then $f(c) = c$.
(b) Is the theorem true if we replace $[a, b]$ with (a, b)? Prove or give a counterexample.
False. Take $f(x) = \frac{x}{2}$ on $(0, 1)$. Then $f(x) : (0, 1) \to (0, 1)$. But if $f(x) = x$, then $\frac{x}{2} = x$ implies $x = 0$ which is not in $(0, 1)$.

3. Use the definition of the derivative to show that

$$(sinx)' = cosx.$$

You may need that $\sin(x + h) = \sin x \cos h + \cos x \sinh$.
Pf: $\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h+\cos x \sin h-\sin x}{h} = \lim_{h\to 0} \left[\sin x \left(\frac{\cos h-1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right]$.
We can then show $\lim_{h\to 0} \frac{\cos h-1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$ using L'Hospital or taylor series, or trig properties. For example, by L'Hospital's Rule we know $\lim_{h\to 0} \frac{\cos h-1}{h} = \lim_{h\to 0} \frac{-\sin h}{1} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\cos h}{1} = 1$. Thus $\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \sin x(0) + \cos x(1) = \cos x$.

4. If

$$C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \ldots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.
Pf: Define the polynomial $p(x) = C_0 x + C_1 \frac{x^2}{2} + \ldots + C_{n-1} \frac{x^n}{n} + C_n \frac{x^{n+1}}{n+1}$. Then $p(0) = 0$ and $p(1) = C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$. The function p is a polynomial so in particular it is continuously differentiable everywhere in $[0, 1]$. Therefore, by the Rolle's Theorem there exists an $x \in (0, 1)$ such that $p'(x) = 0$. Thus $C_0 + C_1 x + \ldots + C_{n-1} x^{n-1} + C_n x^n = 0$.

5. Find an example or explain why one doesn't exist

- A continuous function on $\mathbb{R}$ which is NOT uniformly continuous.
  $f(x) = x^2$
- A uniformly continuous function on $\mathbb{R}$ which is not continuous.
  DNE, all uniformly continuous functions are continuous
- A function where $f'(c) = 0$ but $f(c)$ is not a maximum.
  $f(x) = x^3$
- A continuous function f on $[0, 1]$ such that $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in [0, 1]$
  $f(x) = 3$