21-355 Real Analysis First Exam Fall, 2011

1. Give a precise mathematical definition of:

- A set $E \subset X$ is closed
  E is closed if every limit point of E is a point in E.
- An open cover of a set E in a metric space X
  An open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$
- A point p is an interior point of E
  A point p is an interior point of E if there is a neighborhood n of p such that $N \subset E$.
- A set A is countable
  A set A is countable if and only if there exists a bijection between A and $\mathbb{N}$.

2. State whether the following statements are true or false and justify your answer:

- No infinite set is compact.
  False, $[-5, 5]$ is an infinite set and is compact.
- If a convergent sequence is bounded, then it is monotone.
  False, the sequence $\frac{(-1)^n}{n}$ is bounded and converges to 0, but is not monotone.
- A sequence of rational numbers is convergent if and only if it is a Cauchy sequence in the metric space $\mathbb{Q}$ with $d(x, y) = |x - y|$.
  False, need the completeness of the reals for this statement to hold true. Ex: The sequence $\{a \in \mathbb{Q} : a \geq 0, a^2 < 2\}$. The limit is $\sqrt{2}$ but this isn't in the metric space $\mathbb{Q}$.
- If $s_n \to s$ and $s_n > 0$ for all $n \in \mathbb{N}$ then $s > 0$. False, consider the sequence $s_n = \frac{1}{n}$.

3. Find and prove the limits of:
i) $\frac{\sin n}{n}$(using the definition)
Let $\varepsilon > 0$. Define $N = \frac{1}{\varepsilon}$. Then if $n > N$, $|\frac{\sin n}{n} - 0| \leq |\frac{1}{n}| < \frac{1}{N} = \varepsilon$
ii) $s_1 = 1, s_{n+1} = \frac{1}{3}(s_n + 3)$.
First few terms are $s_1 = 1, s_2 = \frac{4}{3}, s_3 = \frac{13}{9}$, which appear to be increasing and bounded above by 2.
BOUNDED: Clearly bounded below by 0. We claim it is bounded above by 2, prove by induction. Clearly $s_1 < 2$ and assume true for some s_n. Then $s_{n+1} = \frac{1}{3}(s_n + 3) \leq \frac{1}{3}(2 + 3) = \frac{5}{3} < 2$.
INCREASING: Again we prove by induction. Clearly $s_2 - s_1 > 0$. Then $s_{n+1} - s_n = \frac{1}{3}(s_n + 3) - \frac{1}{3}(s_{n-1} + 3) = \frac{1}{s_n - s_{n-1}}$. So if we assume $s_n - s_{n-1} > 0$ then $s_{n+1} - s_n > 0$ so we are done.
LIMIT: We know a limit exists by MCT. Hence let's call it L. So L obeys the relation $L = \frac{1}{3}(L + 3)$ which implies $L = \frac{3}{2}$.

5. If S is a compact subset of $\mathbb{R}$ and T is a subset of S, then T is compact.
Pf: Suppose S is compact, so by Heine Borel it is closed and bounded. Since $T \subset S$, it follows

that T is also bounded. By the hypothesis, T is closed. Thus again by Heine Borel, T is compact.

6. (20 pts.) Every Cauchy sequence is bounded.
Pf: Suppose $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Then we know there exists $N \in \mathbb{R}$ such that $|x_n - x_m| < \varepsilon$ whenever $m, n \geq N$. Thus $|x_n| - |x_m| < |x_n - x_m| < \varepsilon$. In particular, this holds true for $m = N$ which implies $|x_n| < \varepsilon + |x_N|$. Let $M = \max\{|x_1|, |x_2|, \ldots, |x_N|, |x_N| + \varepsilon\}$. Then we have $|x_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded.