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On Certain Properties of Random Apollonian Networks

http://www.math.cmu.edu/~ctsourak/ran.html

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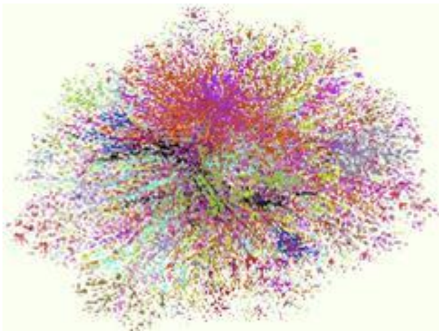
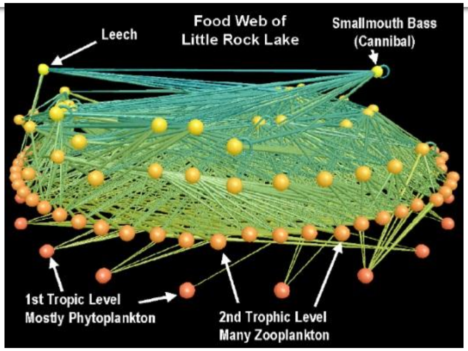
Outline

- **Introduction**
- Degree Distribution
- Diameter
- Highest Degrees
- Eigenvalues
- Open Problems

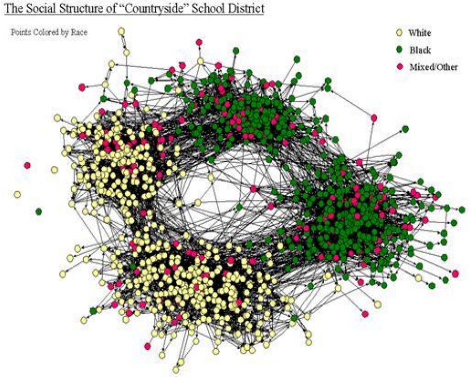
Motivation



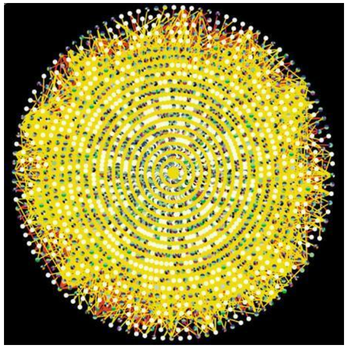
Internet Map
[lumeta.com]



Food Web
[Martinez '91]



Friendship Network
[Moody '01]

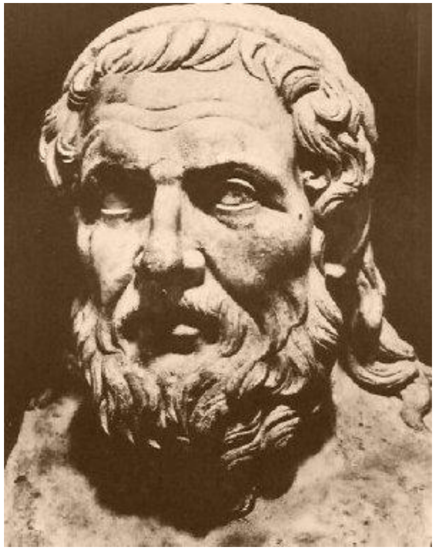


Protein Interactions
[genomebiology.com]

Motivation

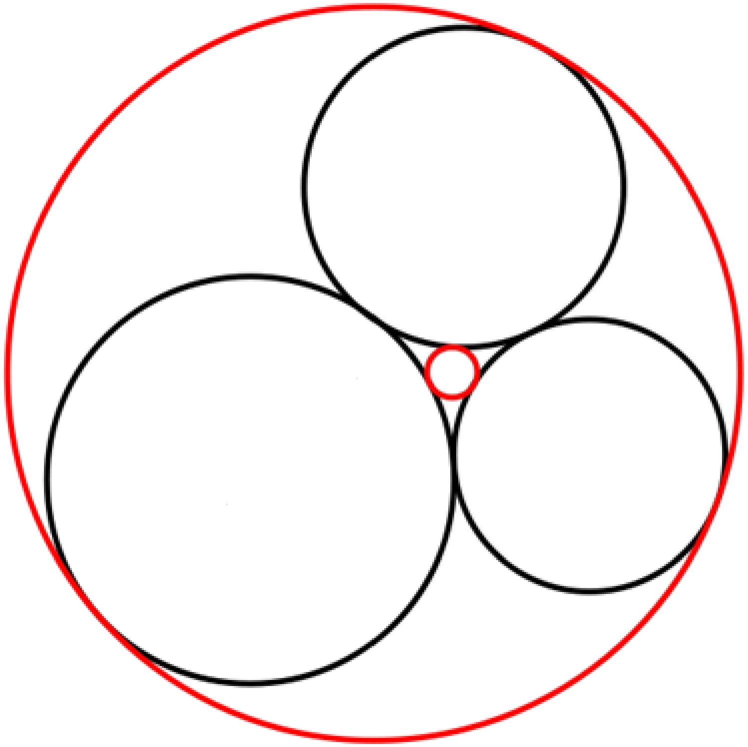
- Modelling "real-world" networks has attracted a lot of attention. Common characteristics include:
  - Skewed degree distributions (e.g., power laws).
  - Large Clustering Coefficients
  - Small diameter
- A popular model for modeling real-world ***planar*** graphs are Random Apollonian Networks.

Problem of Apollonius

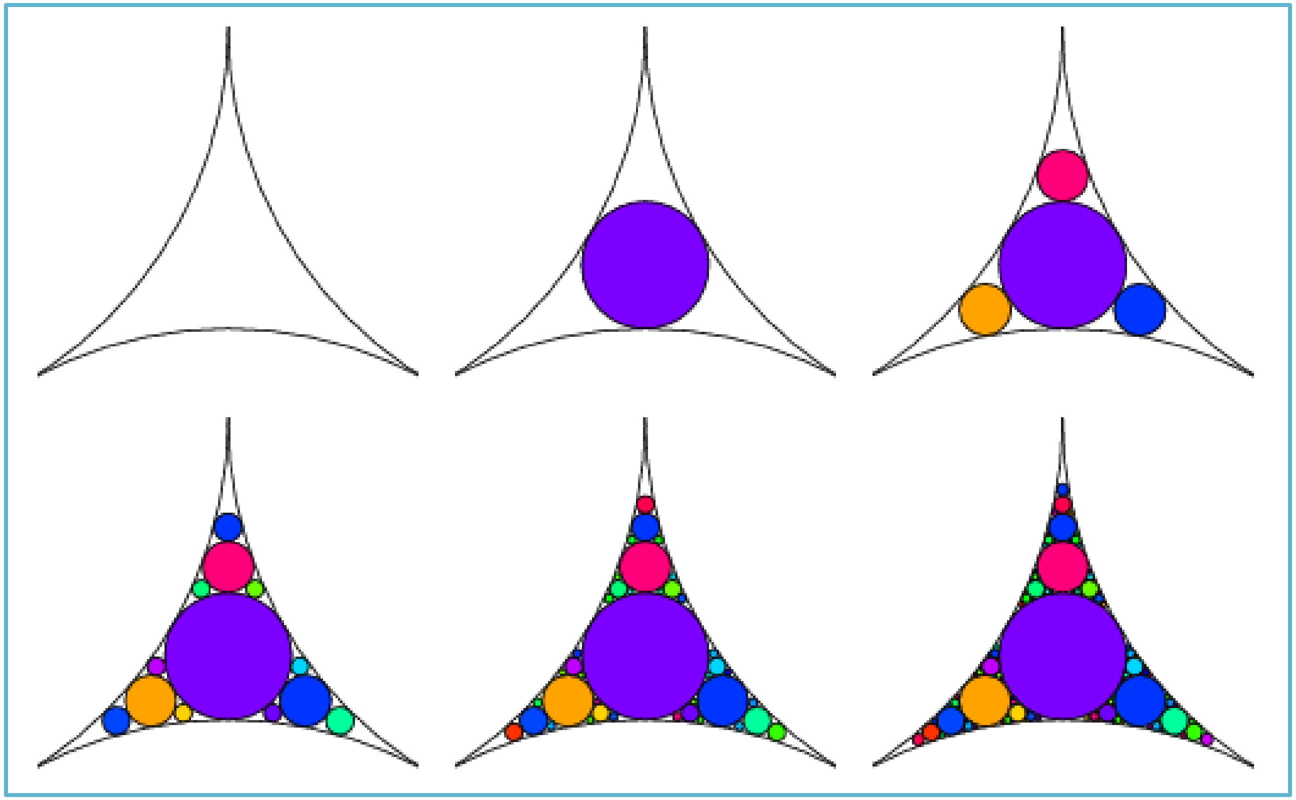


Apollonius
(262-190 BC)

Construct circles that are tangent to three given circles on the plane.

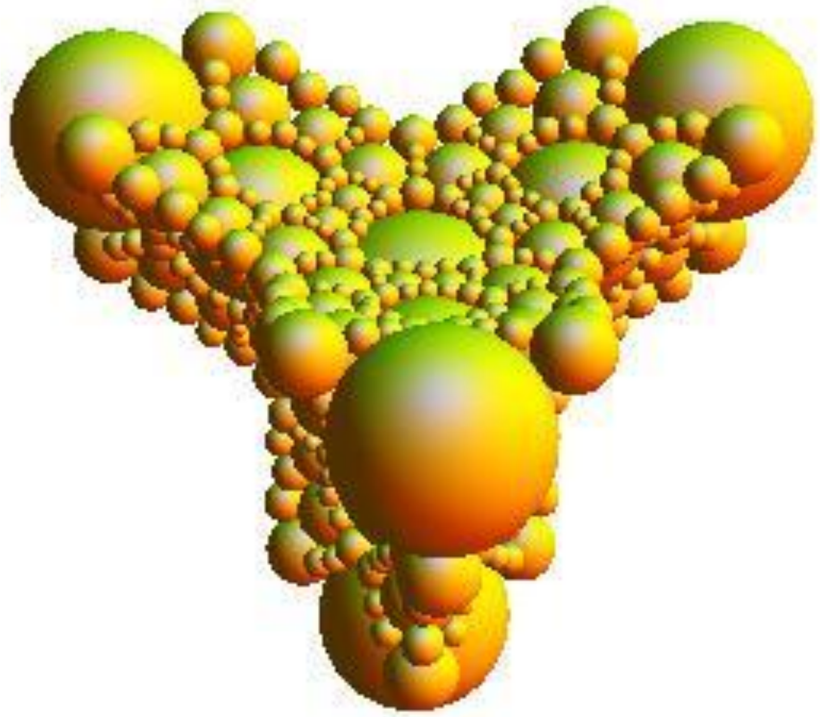


Apollonian Packing



Apollonian Gasket

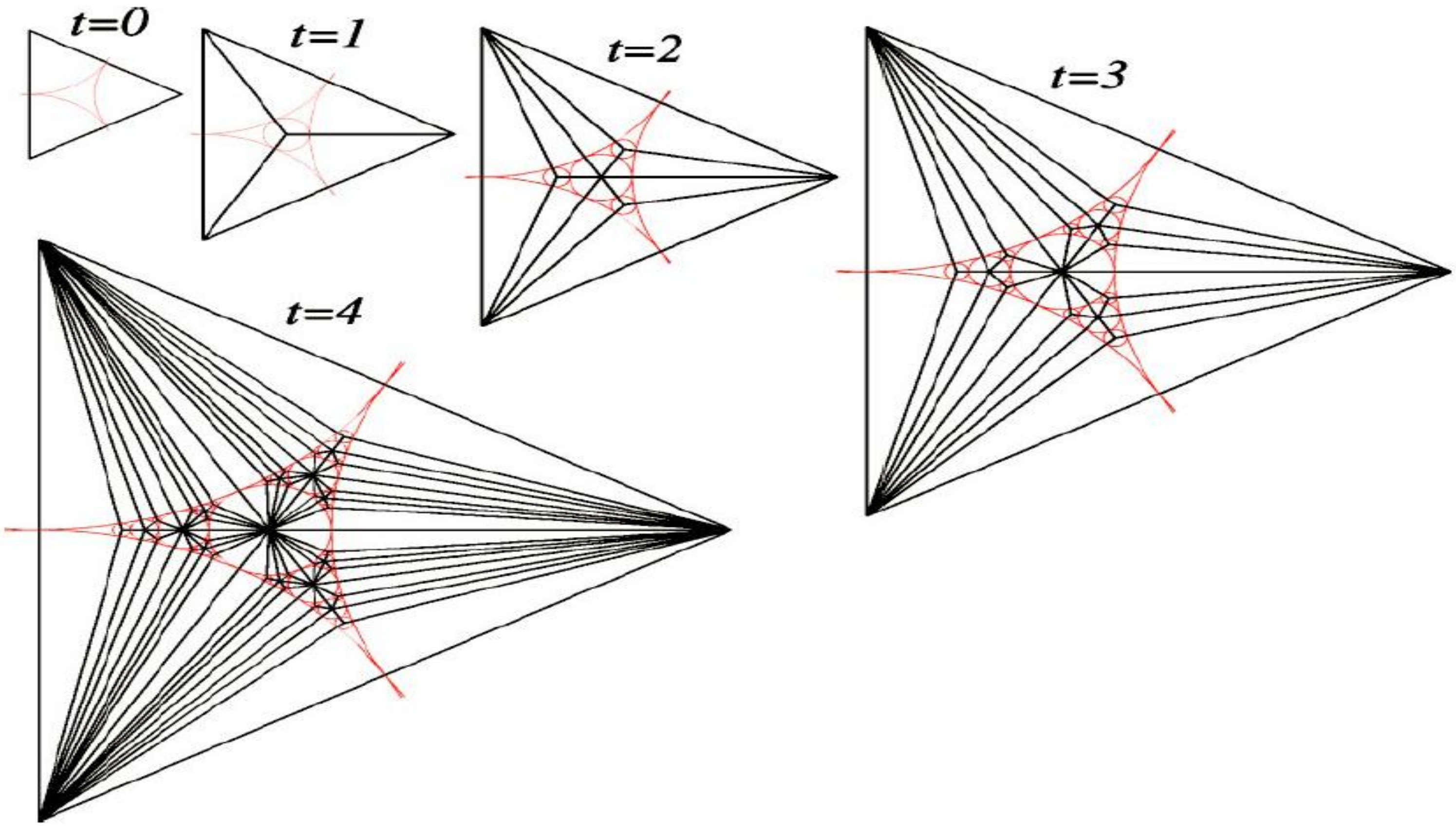
Higher Dimensional Packings



Higher Dimensional (3d) Apollonian Packing. From now on, we shall discuss the 2d case.

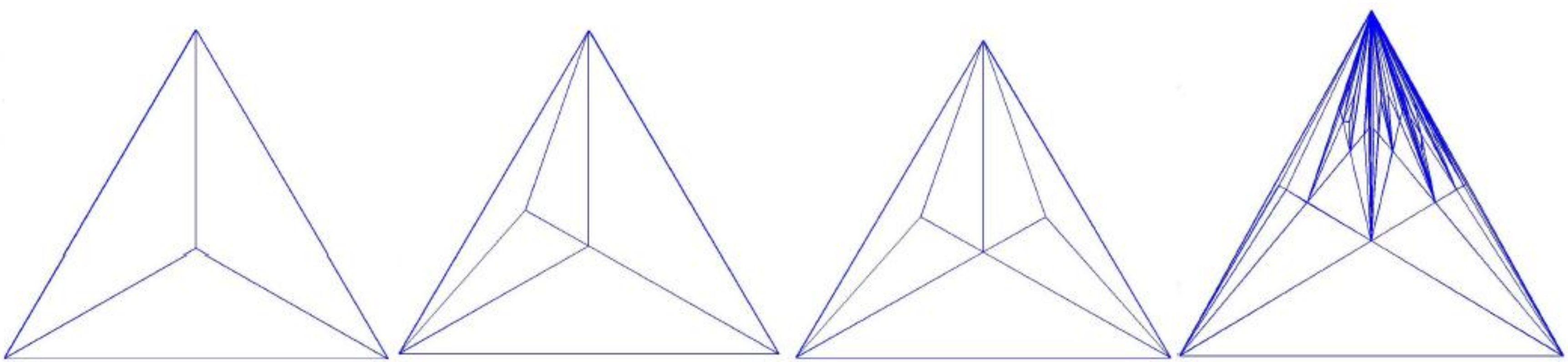
Apollonian Network

- Dual version of Apollonian Packing



Random Apollonian Networks

- Start with a triangle (t=0).
- Until the network reaches the desired size
  - Pick a face F uniformly at random, insert a new vertex in it and connect it with the three vertices of F



(A) $t = 1$ (B) $t = 2$ (C) $t = 3$ (D) $t = 100$

Random Apollonian Networks

For any $t \geq 0$

- Number of vertices n_t =t+3
- Number of vertices m_t=3t+3
- Number of faces F_t=2t+1

Note that a RAN is a maximal planar graph since for any planar graph

$$m_t \leq 3n_t - 6 = 3t + 3$$

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- **Degree Distribution**
- Diameter
- Highest Degrees
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Degree Distribution

- Let $N_k(t)=E[Z_k(t)]$=expected #vertices of degree k at time t. Then:
- $N_3(t+1) = N_3(t) + 1 - \frac{3N_3(t)}{2t+1}$
- $N_k(t+1) = N_k(t)\left(1 - \frac{k}{2t+1}\right) + N_{k-1}(t)\frac{k-1}{2t+1}$

Solving the recurrence results in a power law with "slope 3".

Degree Distribution

- $Z_k(t)$=#of vertices of degree k at time t, $k \geq 3$
- $b_3 = \frac{2}{5}, b_4 = \frac{1}{5}, b_5 = \frac{4}{35}, b_k = \frac{24}{k(k+1)(k+2)} \; k \geq 6$
- For t sufficiently large

$$|E[Z_k(t)] - b_k t| \leq 3.6$$

- Furthermore, for all possible degrees k

$$\mathrm{Prob}(|\mathrm{Z_k(t)} - E[Z_k(t)]| \geq 10\sqrt{tlog(t)}) = o(1)$$

Simulation (10000 vertices, results averaged over 10 runs, 10 smallest degrees shown)

Degree	Theorem	Simulation
3	0.4	0.3982
4	0.2	0.2017
5	0.1143	0.1143
6	0.0714	0.0715
7	0.0476	0.0476
8	0.0333	0.0332
9	0.0242	0.0243
10	0.0182	0.0179
11	0.0140	0.0137
12	0.0110	0.0111

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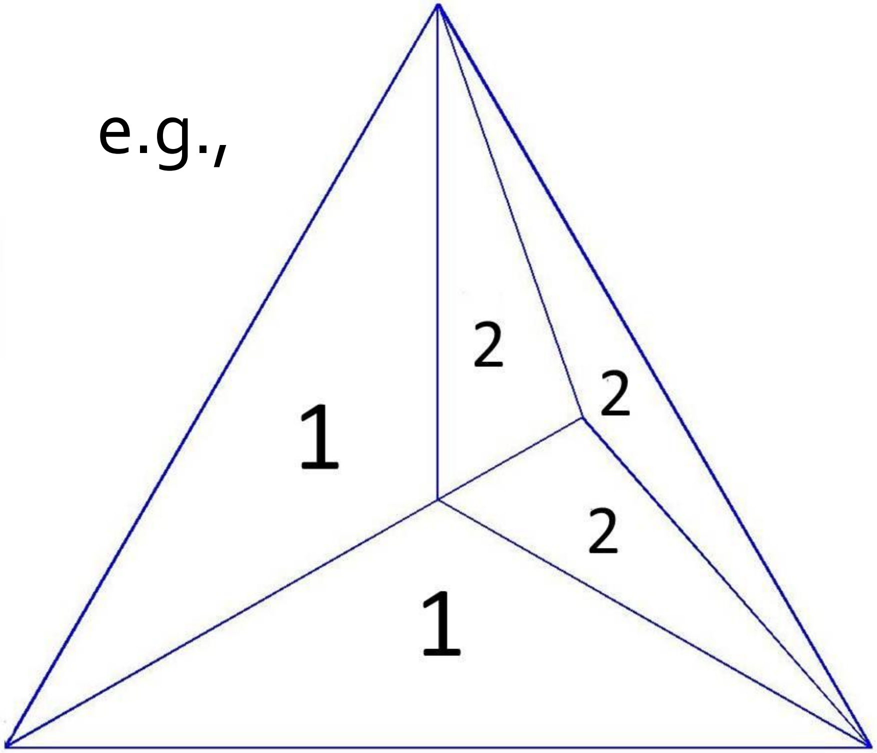
Diameter

Depth of a face (recursively): Let α be the initial face, then depth(α)=1. For a face β created by picking face γ depth(β)=depth(γ)+1.

e.g.,



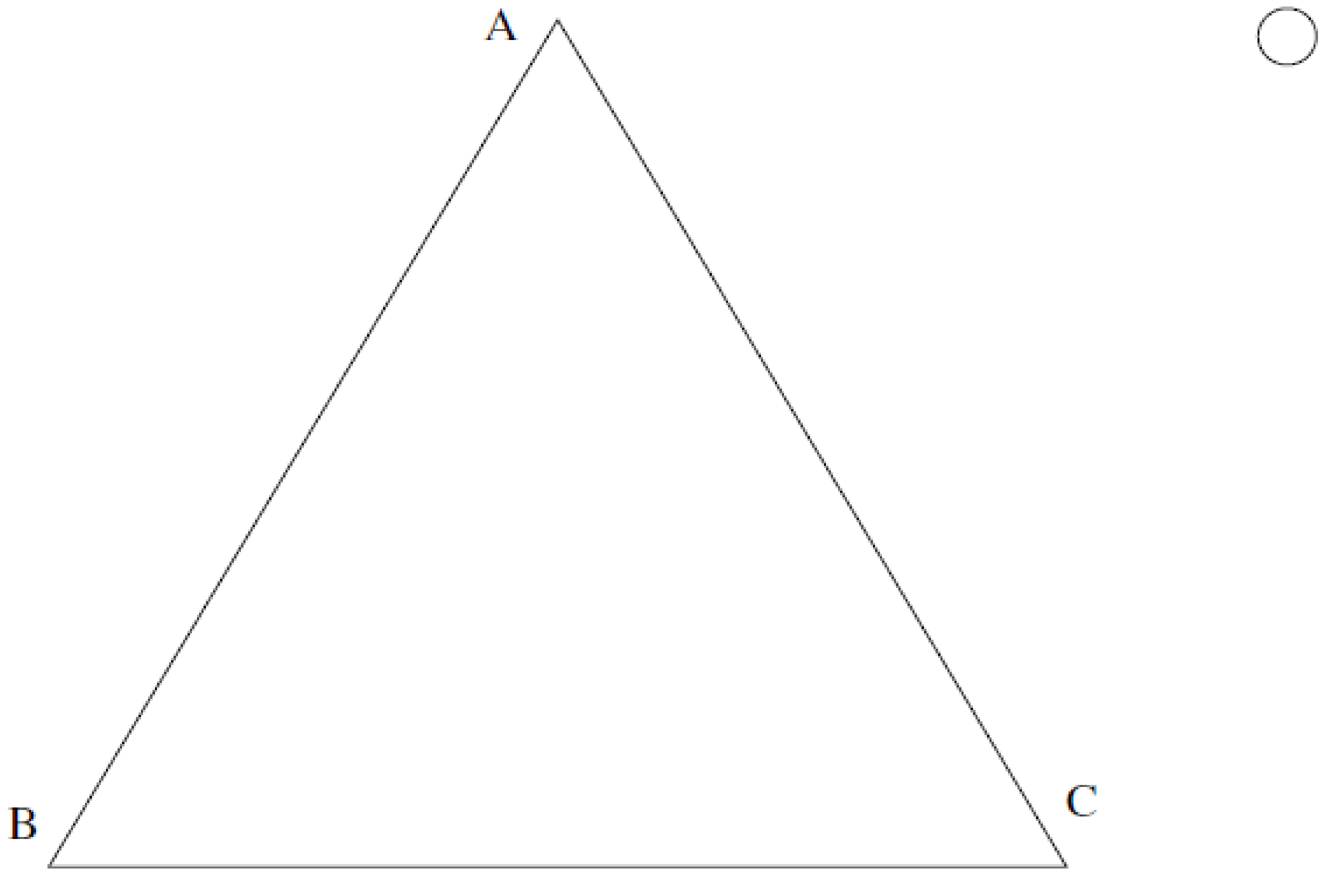
Diameter

- Note that if k* is the maximum depth of a face at time t, then diam(G_t)=O(k*).
- Let F_t(k)=#faces of depth k at time t. Then, $E[F_t(k)]$ is equal to

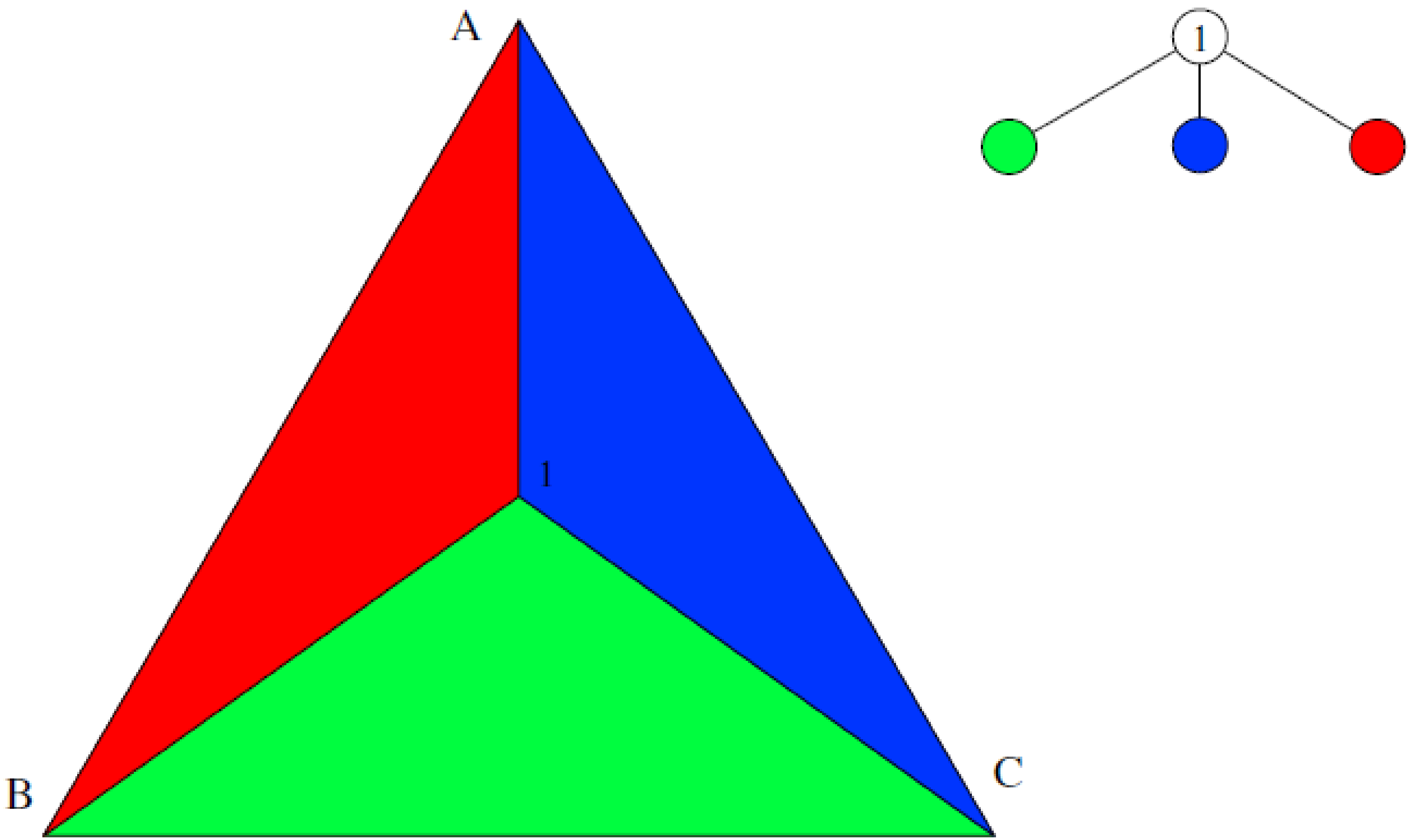
$$\sum_{1 \le t_1 < t_2 < .. < t_k \le t} \prod_{j=1}^{k} \frac{1}{2t_j + 1} \le \frac{1}{k!} \left(\sum_{j=1}^{t} \frac{1}{2j+1} \right)^{t} \le \left(\frac{e log(t)}{2k} \right)^{k+1}$$

Therefore by a first moment argument k*=O(log(t)) *whp*.

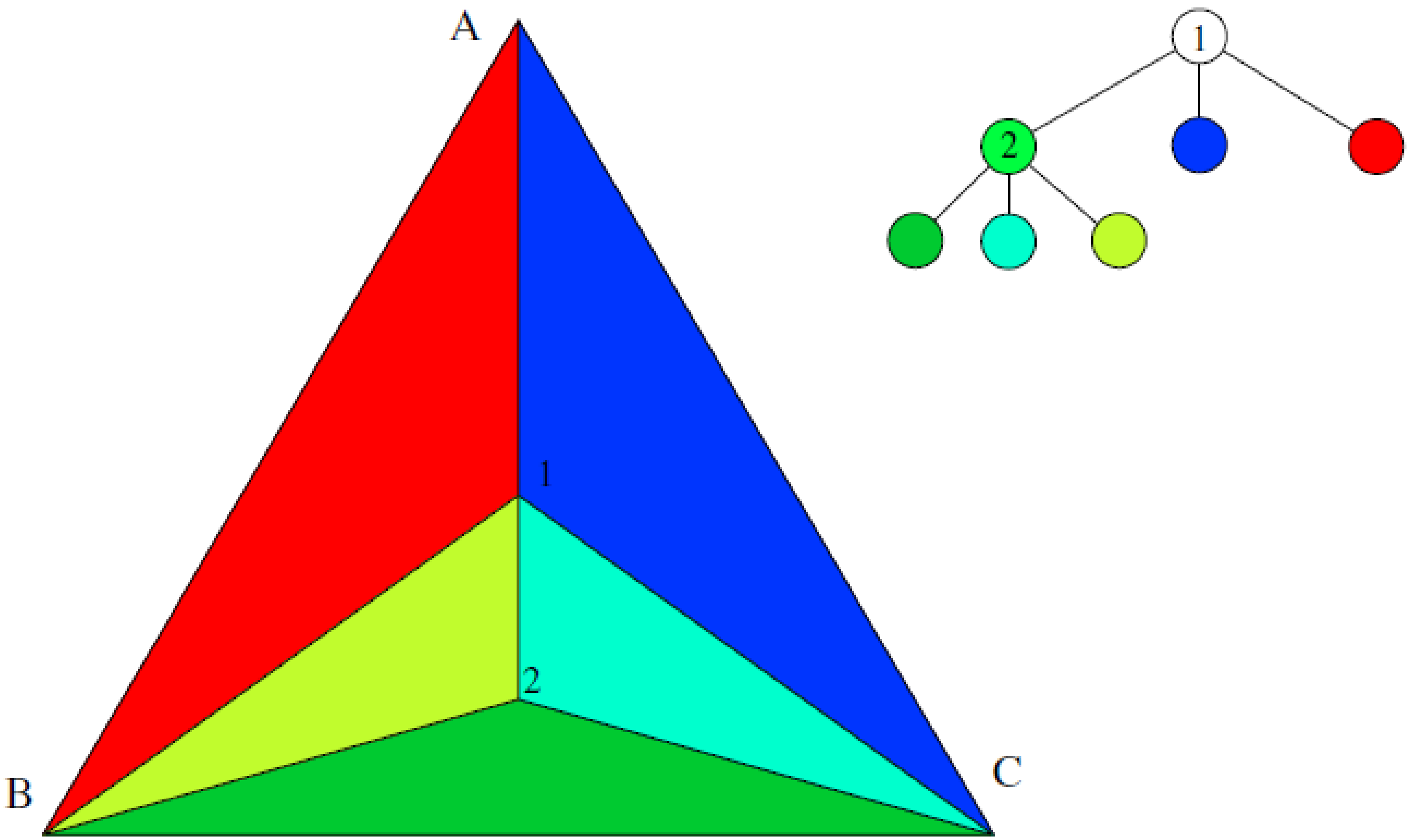
Bijection with random ternary trees



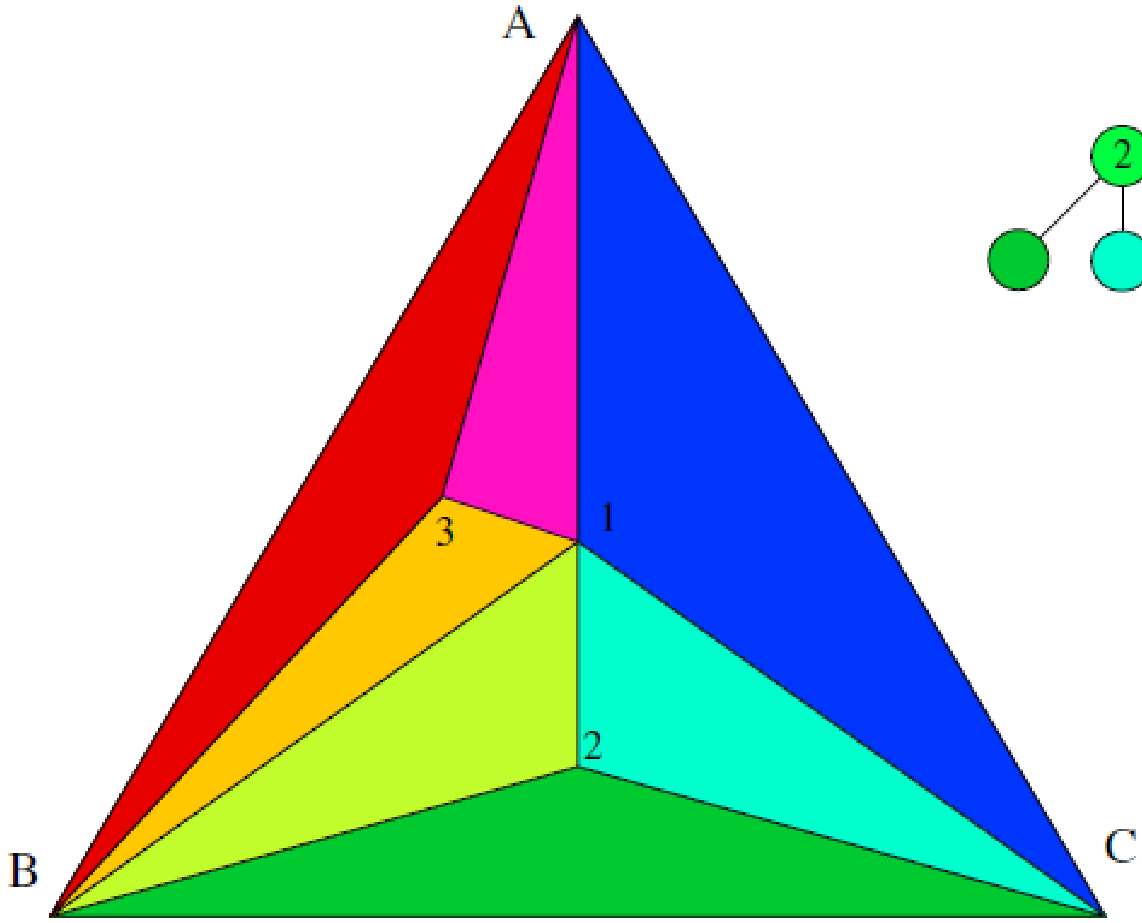
Bijection with random ternary trees

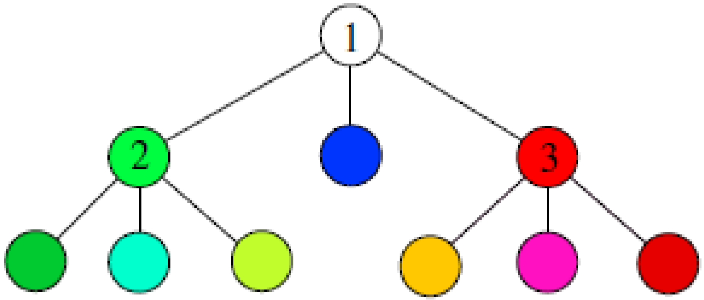


Bijection with random ternary trees



Bijection with random ternary trees





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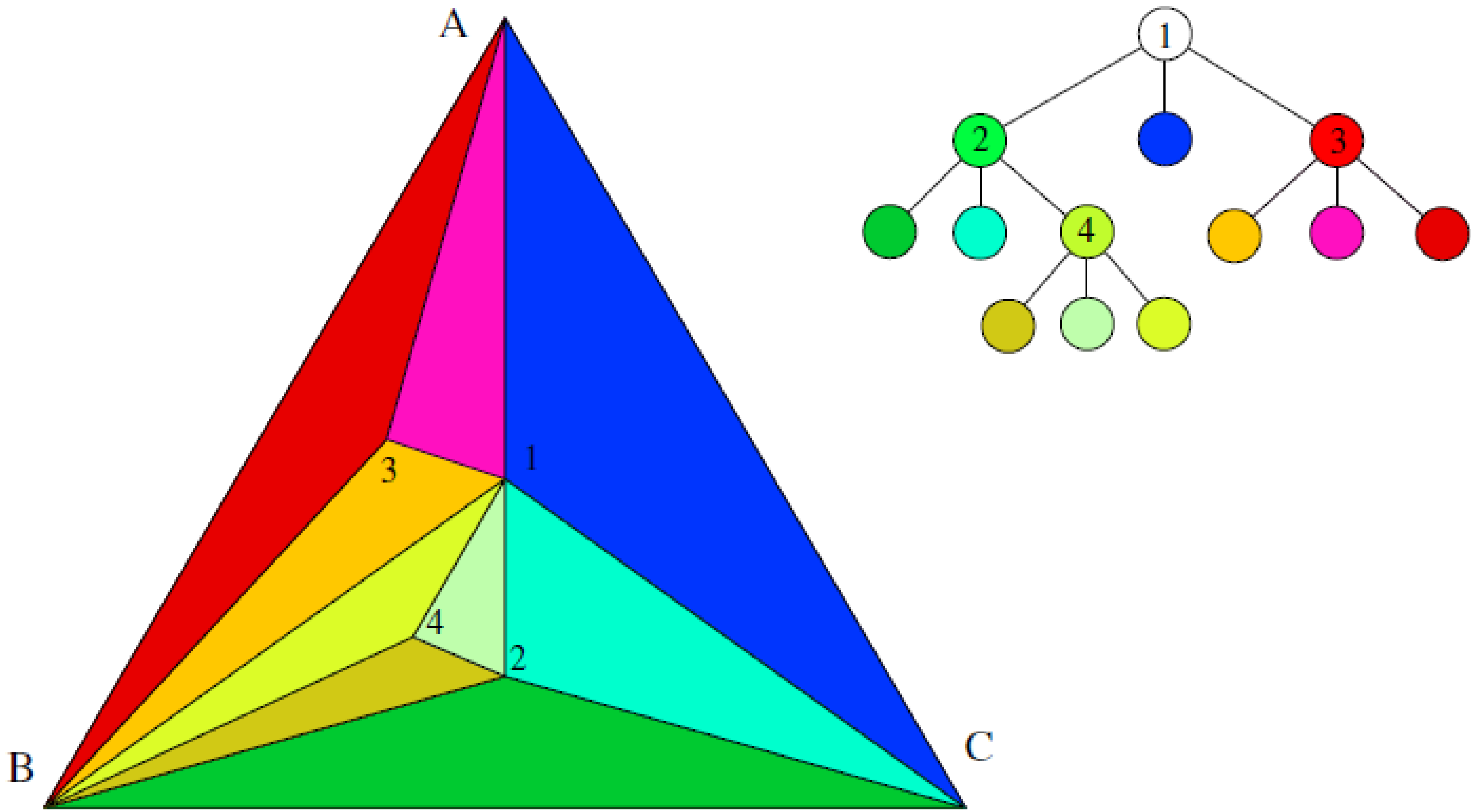
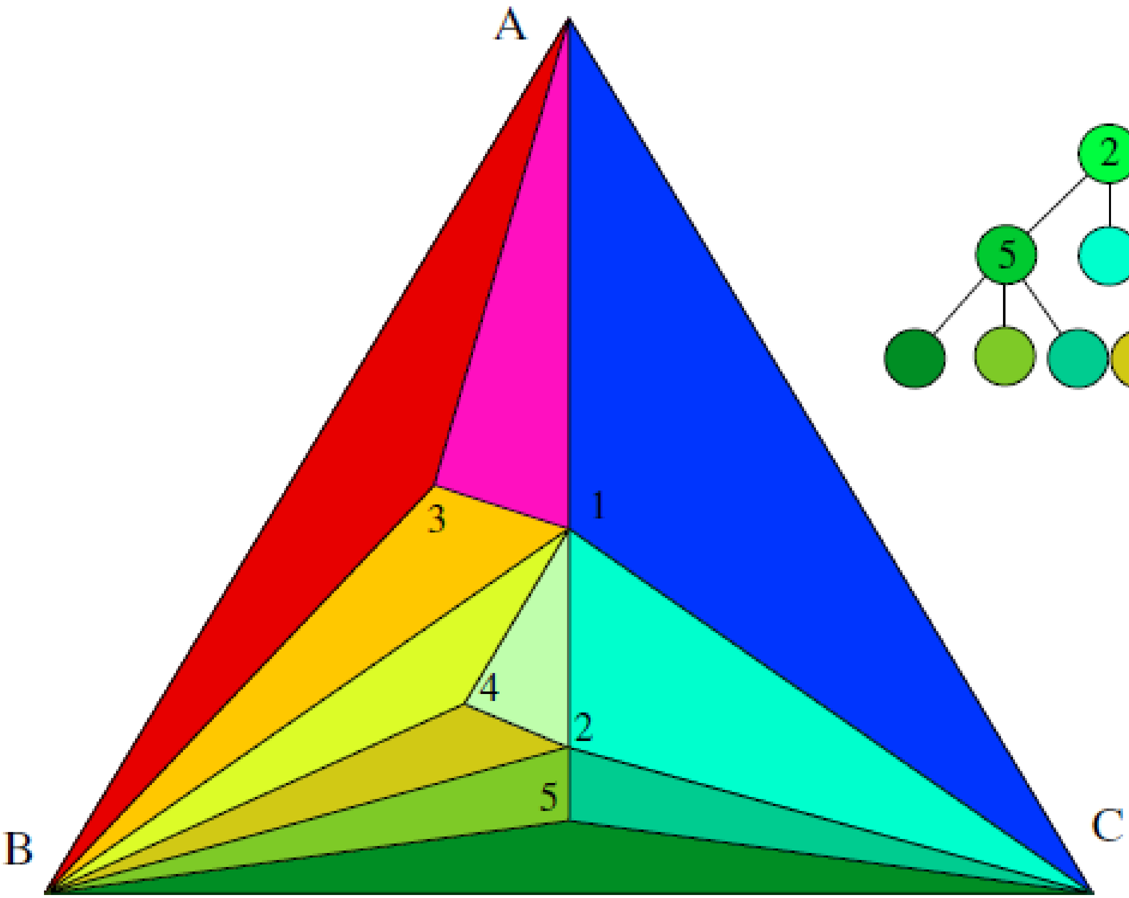
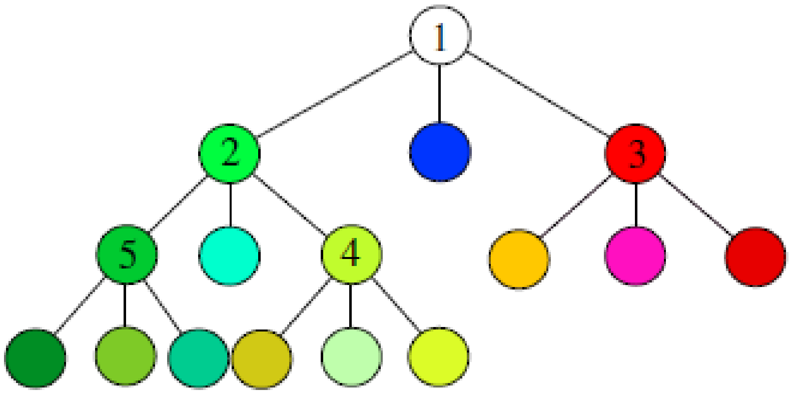


Bijection with random ternary trees





Diameter

Broutin

Devroye

Large Deviations for the Weighted Height of an Extended Class of Trees. Algorithmica 2006

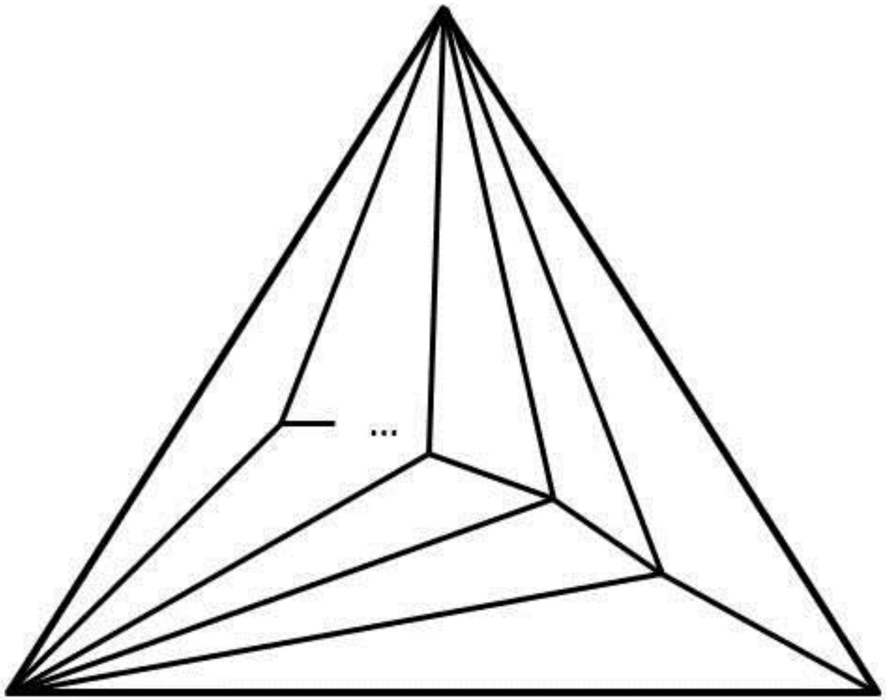
The depth of the random ternary tree T in probability is $\rho/2 \log(t)$ where $1/\rho=\eta$ is the unique solution greater than 1 of the equation $\eta-1-\log(\eta)=\log(3)$.

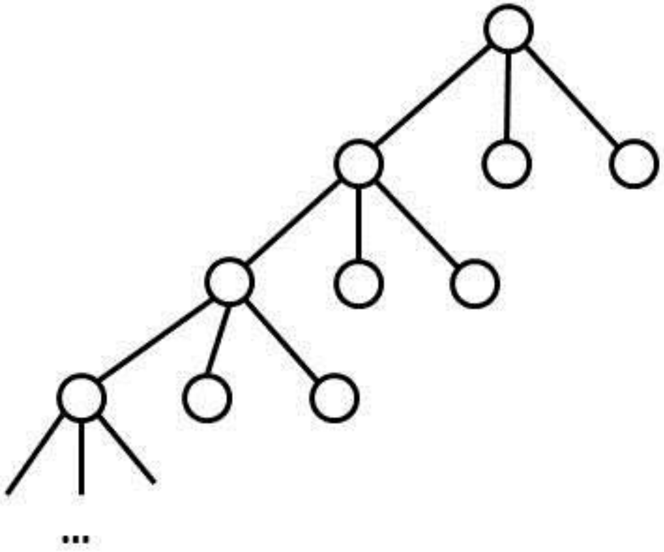
Therefore we obtain an upper bound in probability

$$diam(G_t) \leq \rho \log(t)$$

Diameter

- This cannot be used though to get a lower bound:





Diameter=2,
Depth arbitrarily large

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Highest Degrees, Main Result

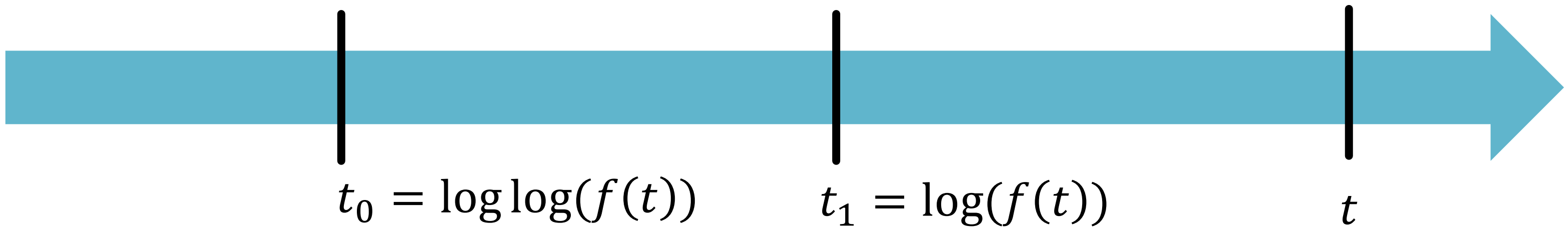
Let $\Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_k$ be the k highest degrees of the RAN G_t where k=O(1). Also let f(t) be a function s.t. $f(t) \xrightarrow[t\to\infty]{} +\infty$. Then *whp*

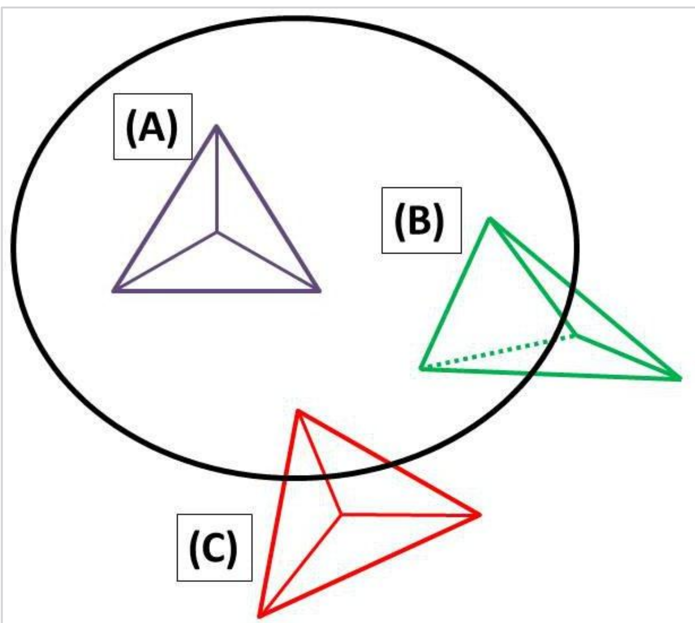
$$\frac{\sqrt{t}}{f(t)} \leq \Delta_1 \leq \sqrt{t}f(t)$$

and for i=2,..,k

$$\frac{\sqrt{t}}{f(t)} \leq \Delta_i \leq \Delta_{i-1} - \frac{\sqrt{t}}{f(t)}$$

Proof techniques





- Break up time in periods
- Create appropriate supernodes according to their age.
- Let Xt be the degree of a supernode. Couple RAN process with a simpler process Y such that

$$X_t \geq Y_t, X_{t_0} = Y_{t_0} = d_0$$

  Upper bound the probability p*(r)=$\Pr(Y_t = d_0 + r)$
- Union bound and k-th moment arguments

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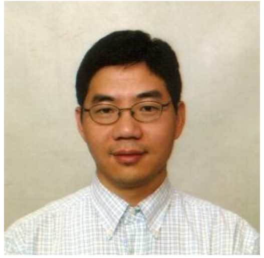
Eigenvalues, Main Result

- Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be the largest k eigenvalues of the adjacency matrix of G_t. Then $\lambda_i = \left(1 \pm o(1)\right)\sqrt{\Delta_{\mathrm{i}}}$ *whp*.
- Proof comes for "free" from our previous theorem due to the work of two groups:

Chung



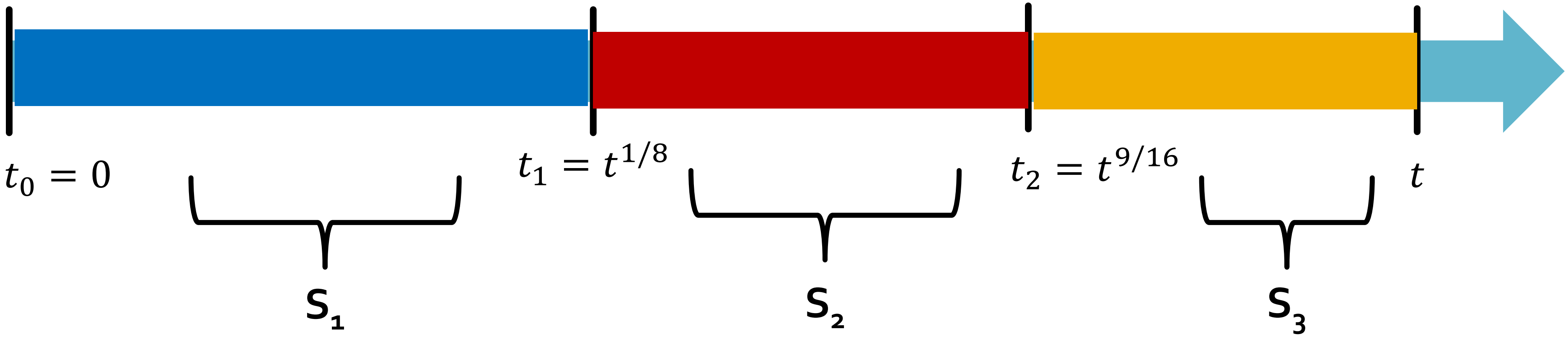
Lu

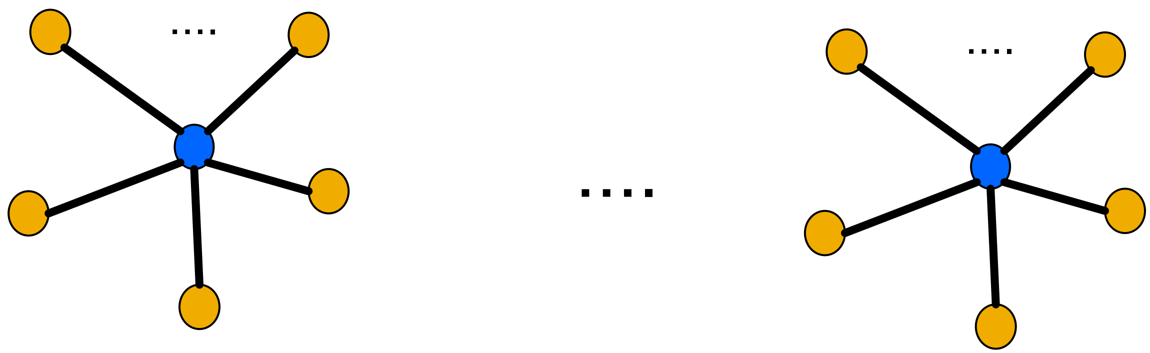
Vu

Mihail

Papadimitriou

Eigenvalues, Proof Sketch

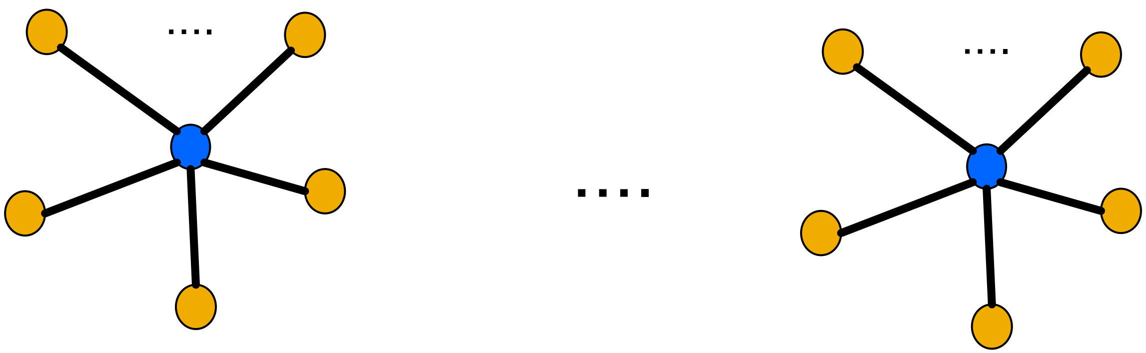




Star forest consisting of edges between S_1 and S_3-S'_3 where S'_3 is the subset of vertices of S_3 with two or more neighbors in S_1.

Eigenvalues, Proof Sketch

- Lemma: $|S'_3| \leq t^{1/6}$
- This lemma allows us to prove that in F



$$\lambda_i(F) = \big(1 - o(1)\big)\sqrt{\Delta_i}$$

Eigenvalues, Proof Sketch

Finally we prove that in H=G-F

$$\lambda_1(\mathrm{H}) = \mathrm{o}(\lambda_\mathrm{k}(\mathrm{F}))$$

<u>Proof Sketch</u>

- First we prove a lemma. For any ε>0 and any f(t) s.t. $f(t) \xrightarrow[t\to\infty]{} +\infty$ the following holds *whp:* for all *s* with $f(t) \le s \le t$ for all vertices $r \le s$ then $d_s(r) \le s^{\varepsilon+\frac{1}{2}} r^{-\frac{1}{2}}$.

Eigenvalues, Proof Sketch

- Consider six induced subgraphs $H_i = H[S_i]$ and $H_{ij} = H(S_i, S_j)$. The following holds:

$$\lambda_1(H) \leq \sum_{i=1}^{3} \lambda_1(H_i) + \sum_{i<j} \lambda_1(H_i, H_j)$$

- Bound each term in the summation using the lemma and the fact that the maximum eigenvalue is bounded by the maximum degree.

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Open Problems

Conductance Φ is at most $t^{-1/2}$.

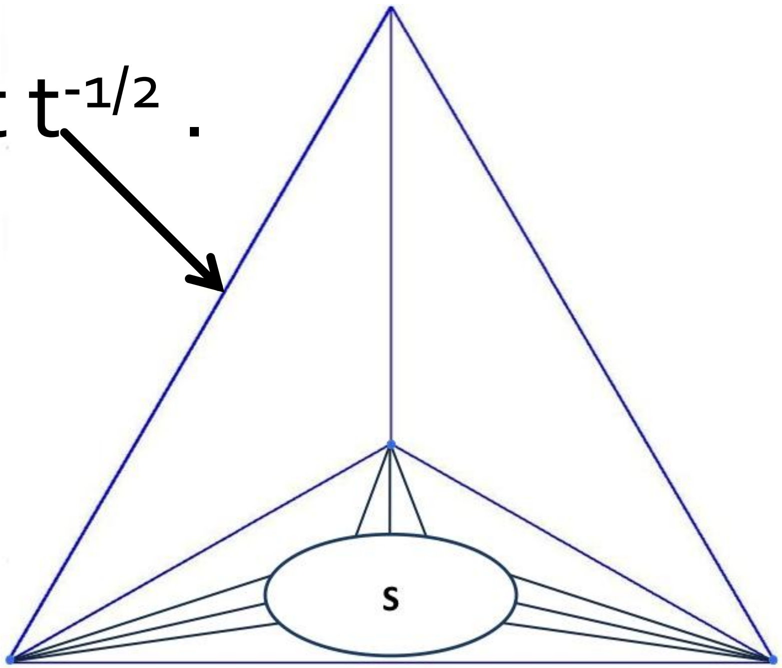
Conjecture: $\Phi = \Theta(t^{-1/2})$

Are RANs Hamiltonian?

Conjecture: No

Length of the longest path?

Conjecture: $\Theta(n)$



Thank you!