Simplex slicing: an asymptotically-sharp lower bound

Colin Tang

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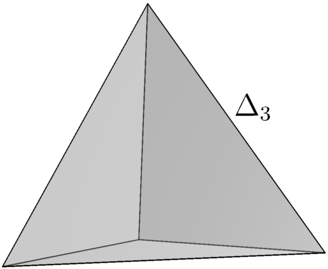
Our goal

Let Δ_n denote the regular n-simplex.



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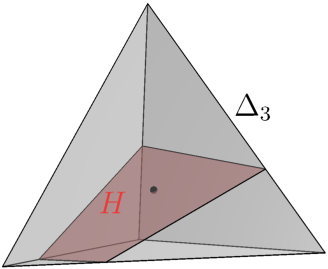

Main question

How may we choose a 1-codimensional hyperplane H passing through the center of Δ_n, so that the volume of the intersection $\mathrm{vol}_{n-1}(\Delta_n \cap H)$ is *minimized*?

Motivation

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- Open problem
- Key to understanding the uniform distribution on a high-dimensional convex body
- Connections to isoperimetry in high dimensions (cf. *KLS conjecture*)

Previous work

A general type of question

Given a specific convex body K, can we identify its minimum central section?

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[1] Hugo Hadwiger. "Gitterperiodische Punktmengen und Isoperimetrie". In: *Monatshefte für Mathematik* 76.5 (1972), pp. 410–418.

[2] Douglas Hensley. "Slicing the Cube in $\mathbb{R}^n$ and Probability (Bounds for the Measure of a Central Cube Slice in $\mathbb{R}^n$ by Probability Methods)". In: *Proceedings of the American Mathematical Society* 73.1 (1979), pp. 95–100.

[3] Keith Ball. "Cube slicing in $\mathbb{R}^n$". In: *Proceedings of the American Mathematical Society* 97.3 (1986), pp. 465–473.

[4] Simon Webb. "Central slices of the regular simplex". In: *Geometriae Dedicata* 61.1 (1996), pp. 19–28.

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Given a specific convex body K, can we identify its minimum central section? Maximum central section?

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- $K = Q_n$, maximal central section identified in [Ball 1986[3]]
- $K = \Delta_n$ (n-dimensional regular simplex), maximal central section identified in [Webb 1996[4]]

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This leaves open the question from the beginning:

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What is the minimum central section of the regular simplex?

[5] Patryk Brzezinski. "Volume estimates for sections of certain convex bodies". In: *Mathematische Nachrichten* 286.17-18 (2013), pp. 1726–1743.

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The minimum central section is the central section $\Delta_n \cap H_{\text{facet}}$ that's parallel to a facet.

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Previous best bound [Brzezinski 2013[5]]

The central section $\Delta_n \cap H_{\mathrm{facet}}$ is within a factor of $\frac{2\sqrt{3}}{e} \approx 1.27$ of the minimum.

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Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024[6]]

The central section $\Delta_n \cap H_{\mathrm{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little o is with respect to the dimension n.)

[6] Colin Tang. "Simplex slicing: an asymptotically-sharp lower bound". In: *Advances in Mathematics* 451 (2024), p. 109784. DOI: https://doi.org/10.1016/j.aim.2024.109784.

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- New: ***moving the contour of integration of a meromorphic function***

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We'll prove this result in the remainder of the presentation.

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Tool: probability distributions

Embed Δ_n into $\mathbb{R}^{n+1}$ via

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} x_1 + x_2 + \cdots + x_{n+1} = 1 \\ x_i \geq 0 \text{ for each } i \end{array} \right\}.$$

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Central sections $\Delta_n \cap H$ correspond to a choice of vector a with

$$\begin{cases} a_1 + a_2 + \cdots + a_{n+1} = 0 \\ a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = 1 \end{cases}$$

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Idea: Instead of Δ_n, consider the density

$$\Phi(x_1, x_2, \dots, x_{n+1}) = \begin{cases} e^{-x_1 - x_2 - \cdots - x_{n+1}} & \text{if each } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Minimum central sections correspond to minimizing $\int_{a^\perp} \Phi \, d\mathcal{H}^n$.

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But Φ is a product measure, so $\int_{a^\perp} \Phi \, d\mathcal{H}^n$ is the density at 0 of the random variable

$$Z_a := a_1 Y_1 + a_2 Y_2 + \cdots + a_{n+1} Y_{n+1}$$

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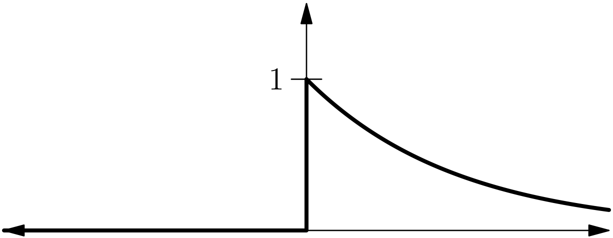

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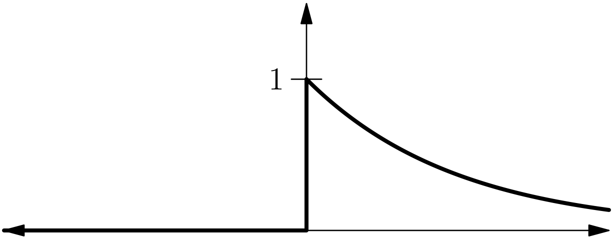

Let $G_a(x)$ denote the density of Z_a, so what we said above is $\int_{a^\perp} \Phi \, d\mathcal{H}^n = G_a(0)$.

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- Let $u \in \mathcal{S}^n$ be arbitrary (the feasible region of u has one fewer constraint than that of a!).
- Define $Z_u \coloneqq u_1(Y_1 - 1) + u_2(Y_2 - 1) + \cdots + u_{n+1}(Y_{n+1} - 1)$.

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- This extends the earlier definition of Z_a since

$$\begin{aligned}
&a_1(Y_1 - 1) + a_2(Y_2 - 1) + \cdots + a_{n+1}(Y_{n+1} - 1) \\
&= a_1 Y_1 + a_2 Y_2 + \cdots + a_{n+1} Y_{n+1} - (a_1 + a_2 + \cdots + a_{n+1}) \\
&= a_1 Y_1 + a_2 Y_2 + \cdots + a_{n+1} Y_{n+1}.
\end{aligned}$$

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$G_u(0) \geq \frac{1}{e}$ for each $u \in \mathcal{S}^n$. Equality achieved if $u = (1) \in \mathcal{S}^0$.

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We lost a bit by expanding the feasible region from $\mathcal{S}^n \cap \mathbf{1}^\perp \ni a$ to $\mathcal{S}^n \ni u$. Indeed, the minimum over u of $G_u(0)$ is exactly $\frac{1}{e}$, but we think the minimum over a of $G_a(0)$ is given by

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$G_{a_{\mathrm{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left(\frac{n}{n+1}\right)^{n-1}$.

But certainly

$$\frac{1}{e} = \min_u G_u(0) \leq \min_a G_a(0) \leq G_{a_{\mathrm{facet}}}(0),$$

and since $G_{a_{\mathrm{facet}}}(0) = \frac{1}{e}(1 + o(1))$, we lost at most a $1 + o(1)$ factor by expanding the feasible region.

Tool: Fourier analysis

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Here, $f_j(x)$ is the density of $u_j(Y_j - 1)$. It's given by $f_j(x) = \frac{1}{|u_j|} f(\frac{x}{u_j} + 1)$ where f is the density of the standard (uncentered) exponential with mean 1:

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Fourier inversion formula, valid if u has at least two nonzero entries:

$$G_u(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{G_u}(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_{j=1}^{n+1} \frac{e^{iu_j t}}{1+iu_j t}\, dt$$

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We wanted to show $\boxed{G_u(0) \geq \frac{1}{e}}$, and this is equivalent to

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Letting $F_u(t) := \prod_{j=1}^{n+1} \frac{e^{iu_j t}}{1+iu_j t}$, we just want to show

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} F_u(t) \, dt \geq \frac{1}{e}}.$$

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I'll spare you the pictures from my first attempt. It really wasn't great.

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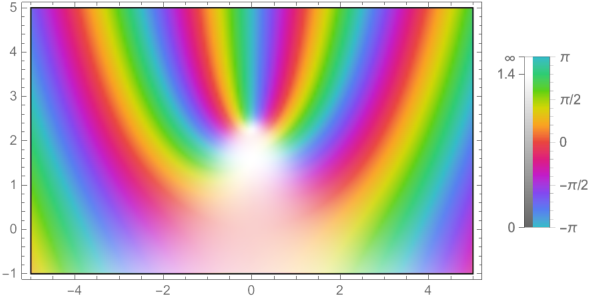
- If we want to estimate the integral $\int_{-\infty}^{+\infty} F_u(t)\,dt$, we can change the contour of integration, from the real line, to a special curve γ_u.
- We will choose γ_u to have the property that F_u is always a positive real number along γ_u.

Tool: moving the contour of integration

Here's a plot of $F_u(t)$ with $u = (\sqrt{0.42}, \sqrt{0.38}, \sqrt{0.20})$:



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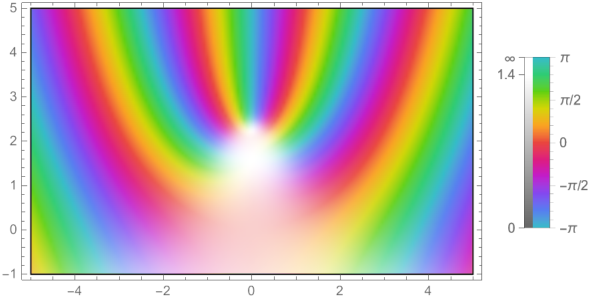

The color denotes the argument of $F_u(t)$. Red means real. Follow the red color, trace out a curve γ_u.

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Black box (basically just the Implicit Function Theorem)

We can always find such a curve γ_u, along which F_u takes positive real values, such that γ_u is $\mathcal{C}^\infty$ and passes through the origin. Moreover, γ_u can be viewed as the graph of an even function $y_u(x)$ in the xy-plane (identified with the complex plane in the usual manner).

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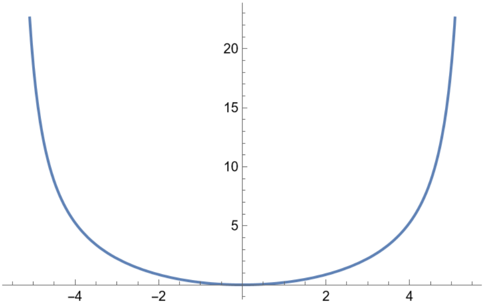
Here's a plot of γ_u with the same u ($u = (\sqrt{0.42}, \sqrt{0.38}, \sqrt{0.20})$):



Tool: moving the contour of integration

Black box (some crude tail bounds)

As long as u has at least two nonzero entries, we have that the integral $\int_{-\infty}^{+\infty} F_u(t)\,dt$ exists and equals $\int_{\gamma_u} F_u(t)\,dt$. Moreover, the integrand $F_u(t)$ is always a positive real number if t is on γ_u.

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This is the part when we actually ***move the contour of integration***.
So we just need to estimate $\int_{\gamma_u} F_u(t)\,dt$.

Differential equations

Recall that $y_u(x)$ is the function whose graph is γ_u.

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So we just need to show $\boxed{\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} \tilde{F}_u(x)\,dx \geq \dfrac{1}{e}}$.

Differential equations

Compute that equality holds if $u = (1) \in \mathcal{S}^0$; i.e.

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as desired.
Let's show the boxed statement. From now on, assume $x > 0$.

Differential equations

Defining property of y_u

$$y_u' = \sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2 + y_u^2)}{x^2 + (\frac{1}{u_j} - y_u)^2} \Bigg/ \sum_{j=1}^{n+1} \frac{x}{x^2 + (\frac{1}{u_j} - y_u)^2}$$

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$$y_u' \leq \frac{-y_u + x^2 + y_u^2}{x}$$

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Corollary

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Using this, we can prove

Black box

$$-y_{(1)} \leq y_u \leq y_{(1)} \text{ for all } x > 0. \qquad (*)$$

Differential equations

Compute

$$\frac{d}{dx}\log\tilde{F}_u(x) = -\frac{\left(\sum_{j=1}^{n+1}\frac{x}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}\right)^2+\left(\sum_{j=1}^{n+1}\frac{-y_u+u_j(x^2+y_u^2)}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}\right)^2}{\sum_{j=1}^{n+1}\frac{x}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}}$$

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Substituting $u = (1)$ yields

$$\frac{d}{dx}\log\tilde{F}_{(1)}(x) = -\frac{x^2+y_{(1)}^2}{x}.$$

Differential equations: Two curious inequalities

Use Cauchy-Schwarz:

$$
\begin{aligned}
\left(\sum_{j=1}^{n+1} \frac{x}{x^2 + \left(\frac{1}{u_j} - y_u\right)^2}\right)^2 &= \left(\sum_{j=1}^{n+1} \frac{(x/u_j)\cdot u_j}{x^2 + \left(\frac{1}{u_j} - y_u\right)^2}\right)^2 \\
&\leq \left(\sum_{j=1}^{n+1} \frac{(x/u_j)^2}{\left(x^2 + \left(\frac{1}{u_j} - y_u\right)^2\right)^2}\right)\left(\sum_{j=1}^{n+1} u_j^2\right) \\
&= \sum_{j=1}^{n+1} \frac{(x/u_j)^2}{\left(x^2 + \left(\frac{1}{u_j} - y_u\right)^2\right)^2}
\end{aligned}
$$

Differential equations: Two curious inequalities

Use Cauchy-Schwarz again:

$$
\begin{aligned}
\left(\sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2+y_u^2)}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}\right)^2 &= \left(\sum_{j=1}^{n+1} \frac{(-y_u/u_j + x^2 + y_u^2)\cdot u_j}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}\right)^2 \\
&\leq \left(\sum_{j=1}^{n+1} \frac{(-y_u/u_j + x^2 + y_u^2)^2}{\left(x^2+\left(\frac{1}{u_j}-y_u\right)^2\right)^2}\right)\left(\sum_{j=1}^{n+1} u_j^2\right) \\
&= \sum_{j=1}^{n+1} \frac{(-y_u/u_j + x^2 + y_u^2)^2}{\left(x^2+\left(\frac{1}{u_j}-y_u\right)^2\right)^2}
\end{aligned}
$$

Differential equations

Putting it together:

$$\frac{d}{dx}\log \tilde{F}_u(x) \geq -\frac{\sum_{j=1}^{n+1} \frac{x^2+y_u^2}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}}{\sum_{j=1}^{n+1} \frac{x}{x^2+\left(\frac{1}{u_j}-y_u\right)^2}}$$

$$= -\frac{x^2+y_u^2}{x}$$

$$\overset{(*)}{\geq} -\frac{x^2+y_{(1)}^2}{x}$$

$$= \frac{d}{dx}\log \tilde{F}_{(1)}(x)$$

which is sufficient to imply $\tilde{F}_u(x) \geq \tilde{F}_{(1)}(x)$, as desired.

Thanks