Math 300 Class 26

Monday 11th March 2019

Definition 1 — *Random variable*
Let $(\Omega, \mathbb{P})$ be a discrete probability space and let E be a set. An **E-valued random variable** on $(\Omega, \mathbb{P})$ is a function $X : \Omega \to E$.

The set E is called the **state space** of X.

We think of X as being a 'variable' element of E depending on the outcome of a random process—if the outcome of the random process is ω, then the value of X is $X(\omega)$.

Example 2
Let X be a real-valued random variable on a discrete probability space $(\Omega, \mathbb{P})$. Express the following events as subsets of Ω.

(a) The event that $X = 0$;

"$X=0$" means $X(\omega)=0$ for the outcome ω of the random process, i.e.

$$\{X=0\} = \{\omega \in \Omega \mid X(\omega)=0\} = X^{-1}[\{0\}]$$

(b) The event that $X \in \mathbb{Z}$;

$$\{X \in \mathbb{Z}\} = \{\omega \in \Omega \mid X(\omega) \in \mathbb{Z}\} = X^{-1}[\mathbb{Z}]$$

(c) The event that $e^{X^2} > 3X+4$.

$$\{e^{X^2} > 3X+4\} = \{\omega \in \Omega \mid e^{X(\omega)^2} > 3X(\omega)+4\} = X^{-1}[\{a \in \mathbb{R} \mid e^{a^2} > 3a+4\}]$$

Exercise 3

A biased coin, which shows heads with probability $0 < p < 1$, is flipped n times, where n is some natural number. Let N be the number of heads that show. Describe a probability space $(\Omega, \mathbb{P})$ which models this random process, give an explicit definition of N as a function $\Omega \to E$ (for an appropriate choice of state space E), and compute $\mathbb{P}\{N = k\}$ for each $k \in E$.

- Define $\Omega = \{0,1\}^n$, where $(i_1, \dots, i_n) \in \{0,1\}^n$ represents the outcome that the k^{th} flip is $\begin{cases} \text{heads} & \text{if } i_k = 1 \\ \text{tails} & \text{if } i_k = 0 \end{cases}$ for each $k \in [n]$.

(Note: the total # heads is $i_1 + i_2 + \dots + i_n$.)

- Define $\mathbb{P}(\{(i_1, \dots, i_n)\}) = p^{\sum_k i_k} (1-p)^{n - \sum_k i_k}$ ← officially we need to check these sum to 1.
- Define $N : \{0,1\}^n \longrightarrow \mathbb{N}$ by

$$N(i_1, \dots, i_n) = \sum_{k=1}^{n} i_k.$$

Then
$$\begin{aligned}
\mathbb{P}(\{N=k\}) &= \mathbb{P}(\{(i_1, \dots, i_n) \mid \text{exactly } k \text{ of } i_1, \dots, i_n \text{ are } 1\}) \\
&= \mathbb{P}\Big[\bigcup_{S \in \binom{[n]}{k}} \{(i_1, \dots, i_n) \mid i_k = 1 \Leftrightarrow k \in S\}\Big] \\
&\overset{(C.A.)}{=} \sum_{S \in \binom{[n]}{k}} p^{|S|} (1-p)^{n-|S|} = \binom{n}{k} p^k (1-p)^{n-k}
\end{aligned}$$

Exercise 4

Let $(\Omega, \mathbb{P})$ be a discrete probability space and let X be a random variable on $(\Omega, \mathbb{P})$. Prove that the events $\{X = e\}$ for $e \in E$ are mutually exclusive (i.e. pairwise disjoint).

Let $e, e' \in E$ & assume $\{X=e\} \cap \{X=e'\}$ is inhabited.

Let $\omega \in \{X=e\} \cap \{X=e'\} = X^{-1}[\{e\}] \cap X^{-1}[\{e'\}]$

Then $\omega \in X^{-1}[\{e\}] \Rightarrow X(\omega) = e$
and $\omega \in X^{-1}[\{e'\}] \Rightarrow X(\omega) = e'$ $\Big\} \Rightarrow e = e'$

So $\{X=e\} = \{X=e'\}$. Hence the events $\{X=e\}$ (for $e \in E$) are pairwise disj't ($\equiv$ mutually exclusive).

Definition 5
Let $(\Omega, \mathbb{P})$ be a discrete probability space and let $X : \Omega \to E$ be a random variable. The **probabilty mass function** of X is the function $f_X : E \to [0,1]$ defined by $f_X(e) = \mathbb{P}\{X = e\}$ for all $e \in E$.

Example 6
Let E be a set and let X be an E-valued random variable on a probability space $(\Omega, \mathbb{P})$. Prove that $X_*\mathbb{P}$ is a probability measure on E, where $(X_*\mathbb{P})(A) = \mathbb{P}\{X \in A\}$ for all $A \subseteq E$.

We need to show that $\sum_{e \in E} (X_* \mathbb{P})(\{e\}) = 1$; then we have that $X_* \mathbb{P}$ is a probability measure on E by the result from last Wednesday.

Now: $\sum_{e \in E} (X_* \mathbb{P})(\{e\})$

$= \sum_{e \in E} \mathbb{P}(\{X = e\})$ by def. of $X_* \mathbb{P}$

$= \sum_{e \in E} \mathbb{P}(X^{-1}[\{e\}])$ by def. of $\{X = e\}$

$= \mathbb{P}\left[\bigcup_{e \in E} X^{-1}[\{e\}]\right]$ by countable additivity

$= \mathbb{P}(\Omega)$ since $X : \Omega \to E$ ($\because \bigcup_{b \in B} f^{-1}[\{b\}] = A$ for any fn $f : A \to B$)

$= 1$

proved on Wednesday.

So $X_* \mathbb{P}$ is a probability measure on E.

Definition 7
Let $(\Omega, \mathbb{P})$ be a probability space. A family of events $\{A_i \mid i \in I\}$ is **mutually independent** if

$$\mathbb{P}\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} \mathbb{P}(A_i)$$

In particular, events $A_1, A_2, \dots, A_n$ are mutually independent if and only if

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \cdots \times \mathbb{P}(A_n)$$

Exercise 8
Let $p \in [0,1]$ and suppose that $X_1, X_2, \dots, X_n$ are $\{0,1\}$-valued with

$$f_{X_k}(i) = \begin{cases} 1-p & \text{if } i = 0 \\ p & \text{if } i = 1 \end{cases}$$

for each $k \in [n]$. Assuming the events $\{X_k = i\}$ are mutually independent for $k \in [n]$ and $i \in \{0,1\}$, prove that the $\{0,1,\dots,n\}$-valued random variable $X = X_1 + \cdots + X_n$ satisfies

$$f_X(r) = \binom{n}{r} p^r (1-p)^{n-r}$$

for each $r \in \{0,1,\dots,n\}$.

Let $r \in \{0, 1, \dots, n\}$. Then

$$\begin{aligned}
f_X(r) &= \mathbb{P}(\{X_1 + X_2 + \cdots + X_n = r\}) \\
&= \mathbb{P}\left(\bigcup_{S \in \binom{[n]}{r}} \left[\bigcap_{k\in S} \{X_k = 1\} \cap \bigcap_{k\in[n]\setminus S} \{X_k = 0\}\right]\right) && \text{since exactly } r \text{ of the } X_k \text{ must be equal to } 1 \\
&= \sum_{S\in\binom{[n]}{r}} \left(\mathbb{P}\left[\bigcap_{k\in S} \{X_k = 1\} \cap \bigcap_{k\in[n]\setminus S} \{X_k = 0\}\right]\right) && \text{C.A.} \\
&= \sum_{S\in\binom{[n]}{r}} \left(\left[\prod_{k\in S} \mathbb{P}(\{X_k = 1\})\right] \times \left[\prod_{k\in[n]\setminus S} \mathbb{P}(\{X_k = 0\})\right]\right) && \text{mutual independence} \\
&= \sum_{S\in\binom{[n]}{r}} p^{|S|} (1-p)^{n-|S|} && \text{def. of } X_k \\
&= \sum_{S\in\binom{[n]}{r}} p^r (1-p)^{n-r} && |S| = r \text{ for all } S \in \binom{[n]}{r} \\
&= \binom{n}{r} p^r (1-p)^{n-r} && \text{all terms in the sum are equal \& the indexing set has } \binom{n}{r} \text{ elements.}
\end{aligned}$$