Math 300 Class 23

Monday 4th March 2019

Definition 1 — *Reflexivity, symmetry and transitivity*
A relation R on a set X is...

- ...**reflexive** if $a R a$ for all $a \in X$;
- ...**symmetric** if, for all $a, b \in X$, if $a R b$, then $b R a$;
- ...**transitive** if, for all $a, b, c \in X$, if $a R b$ and $b R c$, then $a R c$;
- ...an **equivalence relation** if it is reflexive, symmetric and transitive.

Equivalence relations behave in some ways like equality (indeed, equality is reflexive, symmetric and transitive!)—so we will often use symbols like $\sim$ or $\equiv$ or $\approx$, instead of letters like R or S, to denote equivalence relations.

Example 2
Fix $n \in \mathbb{Z}$. Define a relation $\equiv_n$ on $\mathbb{Z}$ by letting $a \equiv_n b$ mean 'n divides $b - a$' for each $a, b \in \mathbb{Z}$. Prove that $\equiv_n$ is an equivalence relation.

- $\underline{\equiv_n \text{ is reflexive}}$: Let $a \in \mathbb{Z}$. Then $a - a = 0 = 0 \times n$
  So n divides $a - a$ $\Rightarrow$ $a \equiv_n a$.

- $\underline{\equiv_n \text{ is symmetric}}$: Let $a, b \in \mathbb{Z}$ and assume $a \equiv_n b$.
  Then $b - a = kn$ for some $k \in \mathbb{Z}$ $\Rightarrow$ $a - b = (-k)n$
  $\Rightarrow$ $b \equiv_n a$.

- $\underline{\equiv_n \text{ is transitive}}$: Let $a, b, c \in \mathbb{Z}$ and assume
  $a \equiv_n b$ and $b \equiv_n c$. Then $b - a = kn$ and
  $c - b = ln$ for some $k, l \in \mathbb{Z}$. So
  $$c - a = (c - b) + (b - a) = kn + ln = (k + l)n$$
  $\Rightarrow$ $a \equiv_n c$. $\square$

Definition 3 — *Equivalence class, quotient*
Let $\sim$ be an equivalence relation on a set X. The **$\sim$-equivalence class** of an element $x \in X$ is the subset $[x]_\sim$ of X defined by
$$[x]_\sim = \{a \in X \mid x \sim a\}$$

If the relation $\sim$ is obvious from context, we may just say 'equivalence class' and write $[x]$, rather than referring to $\sim$ every time.

Example 4
We proved last time that the relation $\sim$ on $\mathbb{R}$ defined by letting $a \sim b$ mean '$a - b \in \mathbb{Q}$' is an equivalence relation. Show that $[0]_\sim = \mathbb{Q}$.

($\subseteq$) Let $x \in [0]_\sim$. Then $0 \sim x$, so $x \sim 0$, and hence $x - 0 = x \in \mathbb{Q}$.

($\supseteq$) Let $x \in \mathbb{Q}$. Then $x - 0 \in \mathbb{Q}$, so $x \sim 0$, and so $0 \sim x \Rightarrow x \in [0]_\sim$.

So $[0]_\sim = \mathbb{Q}$ by double containment.

Example 5
Find the equivalence classes of the integers 0, 1 and 2 with respect to the relation $\equiv_3$ on $\mathbb{Z}$, as defined in Example 2.

For $r \in \{0, 1, 2\}$, and $a \in \mathbb{Z}$, we have

$a \in [r]_{\equiv_3} \Leftrightarrow r \sim a \Leftrightarrow 3$ divides $a - r$

$\Leftrightarrow a - r = 3q$ for some $q \in \mathbb{Z} \Leftrightarrow a = 3q + r$ for some $q \in \mathbb{Z}$

So $[0]_{\equiv_3} = \{a \in \mathbb{Z} \mid a = 3q \text{ for some } q \in \mathbb{Z}\} = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$

$[1]_{\equiv_3} = \{a \in \mathbb{Z} \mid a = 3q + 1 \text{ for some } q \in \mathbb{Z}\} = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$

$[2]_{\equiv_3} = \{a \in \mathbb{Z} \mid a = 3q + 2 \text{ for some } q \in \mathbb{Z}\} = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$

(Note: $[3]_{\equiv_3} = [0]_{\equiv_3}$, $[4]_{\equiv_3} = [1]_{\equiv_3}$, $[5]_{\equiv_3} = [2]_{\equiv_3}$, etc...)

Definition 6 — *Quotient*
The **quotient** of a set X by an equivalence relation $\sim$ on X is the set $X/\sim$ of all $\sim$-equivalence classes of elements of X. That is

$$X/\sim = \{\text{equivalence classes of } \sim\} = \{[x]_\sim \mid x \in X\}$$

The quotient of a set by an equivalence relation *identifies* equivalent elements: the relation $\sim$ on X 'becomes' equality on $X/\sim$, in the sense that

$$\forall a, b \in X,\ a \sim b \Leftrightarrow [a]_\sim = [b]_\sim \quad || \circledast$$

Try to prove this yourself some time.

Example 7
Describe the set $\mathbb{Z}/\equiv_3$.

We saw before that, for all $a \in \mathbb{Z}$,

$[a]_{\equiv_3} = \{$ integers having the same remainder as a when divided by $3\}$

The only possible remainders when divided by 3 are 0, 1 and 2, so

$$\mathbb{Z}/\equiv_3 = \{[0]_{\equiv_3}, [1]_{\equiv_3}, [2]_{\equiv_3}\}$$

Example 8
Prove that $|\mathbb{Z}/\equiv_n| = n$ for all $n > 0$.

As noted above for $n=3$, each $a \in \mathbb{Z}$ leaves a remainder of r when divided by n for a unique $r \in \{0, 1, \dots, n-1\}$; but then $a \in [r]_{\equiv_n}$ for a unique $r \in \{0, 1, \dots, n-1\}$. So

$$\mathbb{Z}/\equiv_n = \{[0]_{\equiv_n}, [1]_{\equiv_n}, \dots, [n-1]_{\equiv_n}\}$$

and the sets in this list are distinct.

$$\Rightarrow |\mathbb{Z}/\equiv_n| = n$$

Definition 9
A **partition** of a set X is a collection $\mathscr{A}$ of inhabited subsets of X such that each $x \in X$ is an element of a unique set $U \in \mathscr{A}$.

The next two results prove that partitions and equivalence relations are essentially the same thing: the equivalence classes give a partition of the set, and each partition of X is the quotient of X by a unique equivalence relation.

Example 10
Let X be a set and let $\sim$ be an equivalence relation on X. Prove that $\mathscr{A} = X/{\sim}$ is a partition of X.

- Each $A \in \mathscr{A}$ is inhabited: Let $A \in \mathscr{A}$. Then $A = [x]$ for some $x \in X$. But then $x \in A$, since $x \sim x$ by reflexivity of $\sim$.
- Each $x \in X$ is an element of a unique $A \in \mathscr{A}$.

(Existence) Let $x \in X$. Then $x \sim x$, so $x \in [x] \in \mathscr{A}$.

(Uniqueness) Let $x \in X$ and assume $A, B \in \mathscr{A}$ with $x \in A$ and $x \in B$. Then:

- $A = [y]$ for some $y \in X$, so $x \sim y$
- $B = [z]$ for some $z \in X$, so $x \sim z$

$\Rightarrow$ $y \sim x$ by symmetry of $\sim$

$\Rightarrow$ $y \sim z$ by transitivity of $\sim$

$\Rightarrow$ $A = [y] \underset{\uparrow \text{ by } (*)}{=} [z] = B$ $\square$

Theorem 11
Let $\mathscr{U}$ be a partition of a set X. There is a unique equivalence relation $\sim$ on X such that $X/{\sim} = \mathscr{A}$. $\square$