Math 300 Class 21

Wednesday 27th February 2019

Definition 1 — *Countably infinite, countable and uncountable sets*
A set X is **countably infinite** if there is a bijection $\mathbb{N} \to X$. A set is **countable** if it is finite or countably infinite, and is **uncountable** if it is not countable.

Exercise 2
Prove that $\mathscr{P}(\mathbb{N})$ is uncountable.

[See Class #20 handout solutions.]

Theorem 3 — *Some facts about countability*

(i) If there is an injection $X \to \mathbb{N}$, then X is countable.

(ii) If there is a surjection $\mathbb{N} \to X$, then X is countable.

(iii) Properties (i) and (ii) remain true if $\mathbb{N}$ is replaced by any countably infinite set.

(iv) If X and Y are countable, then $X \times Y$ is countable.

(v) The union of countably many countable sets is countable.

Example 4

Prove that $\mathbb{Q}$ is countable by defining a surjection from a countable set to $\mathbb{Q}$.

We proved last time that $\mathbb{Z}$ is countable

$\Rightarrow$ $\mathbb{Z}\setminus\{0\}$ is countable by (i)/(iii): since $\mathbb{Z}\setminus\{0\} \subseteq \mathbb{Z}$, the inclusion function $i : \mathbb{Z}\setminus\{0\} \longrightarrow \mathbb{Z}$ defined by $i(n) = n$ for all $n \in \mathbb{Z}\setminus\{0\}$ is injective.

$\Rightarrow$ $\mathbb{Z} \times (\mathbb{Z}\setminus\{0\})$ is countable by (iv)

Define $q : \mathbb{Z} \times (\mathbb{Z}\setminus\{0\}) \longrightarrow \mathbb{Q}$ by

$$q(a,b) = \frac{a}{b} \quad \text{for all } (a,b) \in \mathbb{Z} \times (\mathbb{Z}\setminus\{0\})$$

Then q is surjective: for all $x \in \mathbb{Q}$, we have by definition of $\mathbb{Q}$ that $x = \frac{a}{b} = q(a,b)$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ (i.e. $(a,b) \in \mathbb{Z} \times (\mathbb{Z}\setminus\{0\})$)

$\Rightarrow$ $\mathbb{Q}$ is countable by (ii)/(iii). $\square$

Example 5
Prove that $\mathbb{Q}$ is countable by expressing it as a union of countably many countable sets.

For fixed $b \in \mathbb{Z} \setminus \{0\}$, let $\mathbb{Q}_b = \{ \frac{a}{b} \mid a \in \mathbb{Z} \}$.

- $\mathbb{Z} \setminus \{0\}$ is countable —— we just proved this
- $\mathbb{Q}_b$ is countable for each $b \in \mathbb{Z} \setminus \{0\}$.

  Proof Define $f : \mathbb{Z} \longrightarrow \mathbb{Q}_b$ by $f(a) = \frac{a}{b}$ for each $a \in \mathbb{Z}$. Then f is a bijection $\because$ it has an inverse $g : \mathbb{Q}_b \longrightarrow \mathbb{Z}$ defined by $g(x) = bx$ for all $x \in \mathbb{Q}_b$. Since $\mathbb{Z}$ is countable, we have $\mathbb{Q}_b$ is countable. $\square$

But $\mathbb{Q} = \bigcup_{b \in \mathbb{Z} \setminus \{0\}} \mathbb{Q}_b$: $(\supseteq)$ is immediate since the elements of each $\mathbb{Q}_b$ are rational.

$(\subseteq)$: Let $x \in \mathbb{Q}$. Then $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. But then $b \in \mathbb{Z} \setminus \{0\}$ & $x \in \mathbb{Q}_b$, as required.

So $\mathbb{Q}$ is a union of countably many countable sets

$\Longrightarrow \mathbb{Q}$ is countable. $\square$

Example 6

Prove that $\binom{\mathbb{N}}{2}$, the set of all subsets of $\mathbb{N}$ of size 2, is countably infinite.

We know that $\mathbb{N} \times \mathbb{N}$ is countably infinite

Define $f : \binom{\mathbb{N}}{2} \longrightarrow \mathbb{N} \times \mathbb{N}$ as follows.

Given $U \in \binom{\mathbb{N}}{2}$, write $U = \{a, b\}$ where $a < b$.

Define $f(U) = (a, b)$.

Note f is well defined since each $U \in \binom{\mathbb{N}}{2}$ has a unique expression in list notation as $\{a, b\}$ with $a < b$.

Claim f is injective

Proof Let $U, V \in \binom{\mathbb{N}}{2}$ & assume $f(U) = f(V)$.

Writing $U = \{a, b\}$, $V = \{c, d\}$ for $a, b, c, d \in \mathbb{N}$ with $a < b$ and $c < d$, we have

$$(a, b) = f(U) = f(V) = (c, d)$$

$$\Rightarrow a = c \text{ and } b = d.$$

So $U = \{a, b\} = \{c, d\} = V$.

Since f is injective & $\mathbb{N} \times \mathbb{N}$ is countable, $\binom{\mathbb{N}}{2}$ is countable. (And $\binom{\mathbb{N}}{2}$ is certainly infinite!)

$\square$

Example 6

Prove that $\binom{\mathbb{N}}{2}$, the set of all subsets of $\mathbb{N}$ of size 2, is countably infinite.

Another proof

Define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \binom{\mathbb{N}}{2}$ by

$$f(a,b) = \begin{cases} \{a,b\} & \text{if } a \neq b \\ \{0,1\} & \text{if } a = b \end{cases}$$

← This is just to ensure well-definedness of f.

We know that $\mathbb{N} \times \mathbb{N}$ is countable.

Claim f is surjective

Proof Let $U \in \binom{\mathbb{N}}{2}$. Then $U = \{a,b\}$ for some $a,b \in \mathbb{N}$, and $a \neq b$ since $|U| = 2$

$\Rightarrow U = f(a,b)$.

Since f is surjective & $\mathbb{N} \times \mathbb{N}$ is countable, $\binom{\mathbb{N}}{2}$ is countable. $\square$

Example 6

Prove that $\binom{\mathbb{N}}{2}$, the set of all subsets of $\mathbb{N}$ of size 2, is countably infinite.

Yet another proof

For each $n \in \mathbb{N}$, let $X_n = \{\{n, k\} \mid k > n\}$.

Then:

- X_n is countable for each $n \in \mathbb{N}$. To see this, note that the function $f : \mathbb{N}^{>n} \longrightarrow X_n$ defined by $f(k) = \{n, k\}$ for all $k > n$, is a bijection (it has an inverse $g : X_n \longrightarrow \mathbb{N}^{>n}$ defined by $g(U) = \max U$ for all $U \in X_n$ — that is, $g(\{n, k\}) = k$.)
- $\mathbb{N}$ is countable
- $\binom{\mathbb{N}}{2} = \bigcup_{n \in \mathbb{N}} X_n$: ($\subseteq$) Let $U \in \binom{\mathbb{N}}{2}$. Then $U = \{a, b\}$ for some $a, b \in \mathbb{N}$ with $a < b$. But then $U \in X_a$, so $U \in \bigcup_{n \in \mathbb{N}} X_n$.

  ($\supseteq$) Let $U \in \bigcup_{n \in \mathbb{N}} X_n$. Then $U \in X_n$ for some $n \in \mathbb{N}$, so $U = \{n, k\}$ for some $k > n$. But then $U \subseteq \mathbb{N}$ and $|U| = 2$, so $U \in \binom{\mathbb{N}}{2}$.

Since $\binom{\mathbb{N}}{2}$ is a union of countably many countable sets, it is countable. $\square$