Math 300 Class 18

Wednesday 20th February 2019

Theorem 1 — *Some properties of size*

(a) If Y is finite and there is an injection $X \to Y$, then X is finite and $|X| \leqslant |Y|$;

(b) If X is finite and there is a surjection $X \to Y$, then Y is finite and $|X| \geqslant |Y|$;

(c) If X and Y are finite, then $X \times Y$ is finite and $|X \times Y| = |X| \cdot |Y|$;

(d) If X and Y are finite and $X \cap Y = \varnothing$, then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$. □

Definition 2 — *Binomial coefficients (combinatorial definition)*
Let $n, k \in \mathbb{N}$. The set $\binom{[n]}{k}$ is defined by

$$\binom{[n]}{k} = \{U \subseteq [n] \mid |U| = k\}$$

The **binomial coefficient** $\binom{n}{k}$ is defined by $\binom{n}{k} = \left|\binom{[n]}{k}\right|$.

Example 3
Compute $\binom{3}{k}$ for all $k \in \mathbb{N}$.

$$\binom{[3]}{0} = \{\varnothing\} \Rightarrow \binom{3}{0} = 1$$

$$\binom{[3]}{1} = \{\{1\}, \{2\}, \{3\}\} \Rightarrow \binom{3}{1} = 3$$

$$\binom{[3]}{2} = \{\{1,2\}, \{1,3\}, \{2,3\}\} \Rightarrow \binom{3}{2} = 3$$

$$\binom{[3]}{3} = \{\{1,2,3\}\} \Rightarrow \binom{3}{3} = 1$$

$$\binom{[3]}{k} = \varnothing \text{ for all } k > 3 \Rightarrow \binom{3}{k} = 0 \text{ for all } k > 3.$$

Useful fact: $\binom{n}{k} = \left|\binom{X}{k}\right|$ for *any* set X with $|X| = n$.

Parts (a) and (b) of Theorem 1 combine to give the following useful proof technique.

Strategy (Bijective proof)
In order to prove that finite sets X and Y have the same size, it suffices to find a bijection $X \to Y$.

Example 4
Prove that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ with $k \leqslant n$.

Define $f : \binom{[n]}{k} \longrightarrow \binom{[n]}{n-k}$ by

$f(U) = [n] \setminus U$ for all $U \subseteq [n]$ with $|U| = k$

Note f is well-defined: by Ex 10 from yesterday:

$|[n] \setminus U| = |[n]| - |U| = n - k$

$\Rightarrow [n] \setminus U \in \binom{[n]}{n-k}$ as claimed.

& f is a bijection — it has an inverse

$g : \binom{[n]}{n-k} \longrightarrow \binom{[n]}{k}$

defined by $g(V) = [n] \setminus V$ for all $V \subseteq [n]$ with $|V| = n-k$

(Note: $[n] \setminus ([n] \setminus U) = U$ for all $U \subseteq [n]$
In general, $Y \setminus (Y \setminus X) = X \cap Y$.)

Since there is a bijection $\binom{[n]}{k} \to \binom{[n]}{n-k}$

we have $\left|\binom{[n]}{k}\right| = \left|\binom{[n]}{n-k}\right|$

$\Rightarrow \binom{n}{k} = \binom{n}{n-k}$. $\square$

Definition 5

A **partition** of a finite set X is a family $U_1, U_2, \dots, U_n$ of (inhabited[†]) subsets of X such that:

(i) $\bigcup_{i=1}^{n} U_i = X$; and

(ii) $U_i \cap U_j = \varnothing$ if $i \neq j$ (that is to say that $U_1, \dots, U_n$ are **pairwise disjoint**).

[[†]In the current context, we will additionally allow the sets U_i to be empty.]

Theorem 6 — *Addition principle*

Let X be a finite set and $U_1, U_2, \dots, U_n$ be a partition of X. Then $|X| = \sum_{i=1}^{n} |U_i|$. □

Strategy 7

In order to count the elements of a set X, it suffices to partition X into subsets $U_1, \dots, U_n$ and add up the sizes of the sets in the partition.

Example 8

Prove that, for all $n, k \in \mathbb{N}$, we have $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Let $X = \binom{[n+1]}{k+1}$ and define $\begin{cases} U_1 = \{A \in X \mid n+1 \in A\} \\ U_2 = \{A \in X \mid n+1 \notin A\} \end{cases}$

Then U_1, U_2 is a partition of X $\because$ for all $A \subseteq [n+1]$ we must have $n+1 \in A$ or $n+1 \notin A$, but not both.

<u>Moreover</u>:
- Each $A \in U_1$ is $A' \cup \{n+1\}$ for a unique $A' \subseteq [n]$ with $|A'| = k$ (i.e. $A' \in \binom{[n]}{k}$)
  $\Rightarrow |U_1| = \binom{n}{k}$
- Each $A \in U_2$ is precisely a subset of $[n]$ of size $k+1$ $\Rightarrow |U_2| = \binom{n}{k+1}$

By the addition principle,

$$\binom{n+1}{k+1} = \left|\binom{[n+1]}{k+1}\right| = |U_1| + |U_2| = \binom{n}{k} + \binom{n}{k+1}. \quad \square$$

Theorem 9 — *Multiplication principle*
Fix $m, n \in \mathbb{N}$. Let X be a finite set with $|X| = m$, and for each $a \in X$, let Y_a be a finite set with $|Y_a| = n$. Then

$$|\{(a,b) \mid a \in X,\ b \in Y_a\}| = mn$$

The pair (a,b) is called a **dependent pair**, because the set that b belongs to depends on the value of a. This generalises (by induction!) to sets of dependent n-tuples—the precise statement is ugly.

Strategy 10
Given a finite set X, in order to compute $|X|$, it suffices to devise a step-by-step procedure for uniquely specifying an element of X—each step may depend on the last, but

Example 11

$|A| = k$

Compute the size of the set $X = \{(A,a) \mid A \subseteq [n], a \in A\}$ in two ways:

(a) Specify $(A,a) \in X$ by first choosing A and then choosing a.

- Step 1 Choose $A \subseteq [n]$ with $|A| = k$. There are $\binom{n}{k}$ choices.
- Step 2 Choose $a \in A$. Since $|A| = k$, there are k choices.

By MP, $|X| = \binom{n}{k} \cdot k$.

(b) Specify $(A,a) \in X$ by first choosing a and then choosing A.

- Step 1 Choose $a \in [n]$. There are n choices.
- Step 2 Choose the remaining $k-1$ elements of A from $[n] \setminus \{a\}$. Since $|[n] \setminus \{a\}| = n-1$, there are $\binom{n-1}{k-1}$ choices.

By MP, there are $n \cdot \binom{n-1}{k-1}$ elements in X.

Observe that (a) & (b) imply that $\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$.

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting proofs to show that $|X|$ is equal to both expressions.