

# Math 300 Class 14

Friday 8th February 2019

One of the fundamental properties of the natural numbers is that they are all obtained from 0 by adding 1 some (finite) number of times. This gives rise to convenient strategies for (i) defining expressions involving natural numbers, and (ii) proving statements about such expressions, and about natural numbers more generally.

## Theorem 1 — Recursion theorem

Let  $X$  be a set, let  $a \in X$  and let  $h : \mathbb{N} \times X \rightarrow X$  be a function. There is a unique function  $f : \mathbb{N} \rightarrow X$  such that  $f(0) = a$  and  $f(n+1) = h(n, f(n))$  for all  $n \in \mathbb{N}$ .

## Strategy

In order to specify a function  $f : \mathbb{N} \rightarrow X$ , it suffices to define  $f(0)$  and, for given  $n \in \mathbb{N}$ , assume that  $f(n)$  has been defined and define  $f(n+1)$  in terms of  $n$  and  $f(n)$ .

## Example 2

The expressions  $\sum_{k=1}^n a_k$  and  $\prod_{k=1}^n a_k$ , where  $a_1, a_2, \dots, a_n \in \mathbb{R}$ , are defined recursively by

$$\sum_{k=1}^0 a_k = 0 \quad \text{and} \quad \sum_{k=1}^{n+1} a_k = \left( \sum_{k=1}^n a_k \right) + a_{n+1} \quad \text{for all } n \in \mathbb{N}$$

$$\prod_{k=1}^0 a_k = 1 \quad \text{and} \quad \prod_{k=1}^{n+1} a_k = \left( \prod_{k=1}^n a_k \right) \cdot a_{n+1} \quad \text{for all } n \in \mathbb{N}$$

## Example 3

Give a recursive definition of the factorial  $n!$ , for  $n \in \mathbb{N}$ .

$$0! = 1 \quad \text{and} \quad (n+1)! = (n+1) \cdot n! \\ \text{for all } n \in \mathbb{N}.$$

## Example 4

Given  $n, k \in \mathbb{N}$ , the binomial coefficient  $\binom{n}{k}$  is defined by *double recursion* as follows:

$$\binom{n}{0} = 1 \quad \text{for all } n \in \mathbb{N}, \quad \binom{0}{k+1} = 0 \quad \text{for all } k \in \mathbb{N}, \quad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \quad \text{for all } n, k \in \mathbb{N}$$

**Theorem 5** — *Weak induction principle*

Let  $p(n)$  be a logical formula with free variable  $n \in \mathbb{N}$ . If

- (i)  $p(0)$  is true; and
- (ii)  $\forall n \in \mathbb{N}, (p(n) \Rightarrow p(n+1))$  is true;

then  $p(n)$  is true for all  $n \in \mathbb{N}$ .

**Strategy** (Proof by (weak) induction)

In order to prove a proposition of the form  $\forall n \in \mathbb{N}, p(n)$ , it suffices to prove that  $p(0)$  is true and that, for all  $n \in \mathbb{N}$ , if  $p(n)$  is true, then  $p(n+1)$  is true.

Some terminology:

- The proof of  $p(0)$  is called the **base case**;
- The proof of  $\forall n \in \mathbb{N}, (p(n) \Rightarrow p(n+1))$  is called the **induction step**;
- In the induction step, the assumption  $p(n)$  is called the **induction hypothesis**;
- In the induction step, the proposition  $p(n+1)$  is called the **induction goal**.

Some advice:

- It should be clear to the reader that you are choosing to prove a statement by induction.
- The reader should be able to identify the base case, induction step and induction hypothesis in your proof (either because they're explicitly labelled as such, or from context).
- It helps to write down the induction goal so that you know what you're aiming at.
- Any variables you use should be correctly quantified or introduced.

Some common errors and proof-writing taboos:

- Confusing a logical formula  $p(n)$  (where each  $p(n)$  is a *statement* that is either true or false) with a function  $p: \mathbb{N} \rightarrow \mathbb{N}$  (where each  $p(n)$  is a natural number).
- Accidentally proving  $\forall n \in \mathbb{N}, (p(n+1) \Rightarrow p(n))$  in the induction step. [If you find yourself *deriving* the induction hypothesis and saying something like '... which is true', that should be a red flag that this is what you did.]
- Feeling so comfortable with the heavily structured proof strategy that you forget to use words.

### Example 6

Prove by induction that  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$  for all  $n \in \mathbb{N}$ .

We proceed by induction on  $n$ .

(Base case) We need to prove  $\sum_{k=1}^0 k = \frac{0(0+1)}{2}$ .

This is true since  $\underbrace{\sum_{k=1}^0 k}_{\text{by def. of } \Sigma} = 0$  and  $\frac{0(0+1)}{2} = 0$ .

(Induction step) Fix  $n \in \mathbb{N}$  and assume  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ .

We want to show  $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$ .

$$\begin{aligned} \text{Well: } \sum_{k=1}^{n+1} k &= \left( \sum_{k=1}^n k \right) + (n+1) && \text{by def. of } \Sigma \\ &= \frac{n(n+1)}{2} + (n+1) && \text{by IH} \\ &= (n+1) \left( \frac{n}{2} + 1 \right) && \text{factoring out } (n+1) \\ &= \frac{(n+1)(n+2)}{2} && \text{rearranging.} \end{aligned}$$

This is as required, so the result follows by induction. □

### Example 7

Prove by induction that  $n^3 - n$  is divisible by 3 for all  $n \in \mathbb{N}$ .

(Base case)  $0^3 - 0 = 0 = 3 \times 0$ , so  $0^3 - 0$  is divisible by 3.

(Induction step) Fix  $n \in \mathbb{N}$  and assume  $n^3 - n$  is divisible by 3. Then  $n^3 - n = 3k$  for some  $k \in \mathbb{Z}$ .

So:

$$\begin{aligned}(n+1)^3 - (n+1) &= (n^3 + 3n^2 + 3n + 1) - (n+1) && \text{(expanding)} \\ &= (n^3 - n) + 3n^2 + 3n && \text{(rearranging)} \\ &= 3k + 3n^2 + 3n && \text{(by IH)} \\ &= 3(k + n^2 + n) && \text{(factorising)}\end{aligned}$$

Since  $k + n^2 + n \in \mathbb{Z}$ , it follows that  $(n+1)^3 - (n+1)$  is divisible by 3, as required.

So the result follows by induction.  $\square$

### Different base cases

We might want to prove something is true for all  $n \geq n_0$ , where  $n_0$  is some integer other than 0. But this is the same as proving  $p(n+n_0)$  is true for all  $n \in \mathbb{N}$ . Replacing  $p(n)$  by  $p(n+n_0)$  in Theorem 5 reveals the following strategy for proving results by induction with a base case other than 0.

**Strategy** (Proof by (weak) induction, arbitrary base case)

Let  $n_0 \in \mathbb{Z}$ . In order to prove a proposition of the form  $\forall n \geq n_0, p(n)$ , it suffices to prove that  $p(n_0)$  is true and that, for all  $n \geq n_0$ , if  $p(n)$  is true, then  $p(n+1)$  is true.

**Example 8**Find all natural numbers  $n \in \mathbb{N}$  such that  $3n+1 \leq 2^n$ .

$n$	0	1	2	3	4	5	6	...
$3n+1$	1	4	7	10	13	16	19	...
$2^n$	1	2	4	8	16	32	64	...
	✓	x	x	x	✓	✓	✓	...

From the above table we see  $3n+1 \leq 2^n$  is true when  $n=0$  and false when  $n \in \{1, 2, 3\}$ . We prove  $3n+1 \leq 2^n$  is true for all  $n \geq 4$  by induction on  $n$ .

(BC)  $3 \times 4 + 1 = 13 \leq 16 = 2^4$

(IS) Fix  $n \geq 4$  and assume  $3n+1 \leq 2^n$ .

We want to show  $3(n+1)+1 \leq 2^{n+1}$

Well:

$$3(n+1)+1 = (3n+1) + 3$$

$$\leq 2^n + 3$$

$$\leq 2^n + 2^n$$

$$= 2 \cdot 2^n$$

$$= 2^{n+1}$$

rearranging

by IH

since  $3 < 2^4 \leq 2^n$  ( $\because n \geq 4$ )

rearranging

rearranging

as required.

So the result follows by induction.  $\square$