Math 300 Class 13

Wednesday 6th February 2019

Definition 1
A function $f : X \to Y$ is...

- ...**injective** if, for all $a, b \in X$, if $f(a) = f(b)$, then $a = b$;
- ...**surjective** if, for all $c \in Y$, there exists $a \in X$ such that $f(a) = c$;
- ...**bijective** if it is injective and surjective.

For the next couple of examples, it will be helpful to remark that, for a function $f : X \to Y$ and elements $x \in X$ and $y \in Y$, we have

$$x \in f^{-1}[\{y\}] \quad \Leftrightarrow \quad f(x) \in \{y\} \quad \Leftrightarrow \quad f(x) = y$$

That is, $\boxed{f^{-1}[\{y\}] = \{x \in X \mid f(x) = y\}}$.

Example 2
Let $f : X \to Y$ be a function. Prove that f is surjective if and only if, for all $y \in Y$, the set $f^{-1}[\{y\}]$ has *at least* one element.

($\Rightarrow$) Assume f is surjective, and let $y \in Y$. Then there is some $x \in X$ s.t. $f(x) = y$, since f is surjective. But then $x \in f^{-1}[\{y\}]$, so $f^{-1}[\{y\}]$ has at least one element.

($\Leftarrow$) Assume $\forall y \in Y$, $f^{-1}[\{y\}]$ has at least one element. To see that f is surjective, let $y \in Y$. Take $x \in f^{-1}[\{y\}]$. Then $x \in X$ and $f(x) = y$. So f is surjective.

Example 3

Let $f : X \to Y$ be a function. Prove that f is injective if and only if, for all $y \in Y$, the set $f^{-1}[\{y\}]$ has *at most* one element.

($\Rightarrow$) Assume f is injective. Let $y \in Y$ and assume $a, b \in f^{-1}[\{y\}]$. We need to show $a = b$.

But $a, b \in f^{-1}[\{y\}] \Rightarrow f(a) = y$ & $f(b) = y$

$\Rightarrow f(a) = f(b)$

$\Rightarrow a = b$ since f is injective.

($\Leftarrow$) Assume $f^{-1}[\{y\}]$ has at most one element for each $y \in Y$.

Let $a, b \in X$ & assume $f(a) = f(b)$.

We need to prove $a = b$.

So define $y = f(a)$. Then $a \in f^{-1}[\{y\}]$

and $y = f(a) = f(b) \Rightarrow b \in f^{-1}[\{y\}]$

Since $a, b \in f^{-1}[\{y\}]$ and $f^{-1}[\{y\}]$ has at most one element, we have $a = b$, as required.

Example 4

Prove that there does not exist a surjection $[2] \to [3]$.

Let $f : [2] \to [3]$ be any function.

The set $f[[2]] = \{f(1), f(2)\} \subseteq [3]$ has at most 2 elements, so $\exists$ some $k \in [3] \setminus \{f(1), f(2)\}$.

But then $k \neq f(1)$ and $k \neq f(2)$, so f is not surjective.

Definition 5
An **inverse** for a function $f : X \to Y$ is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Theorem 6
A function $f : X \to Y$ is a bijection if and only if it has an inverse.

Proof
($\Rightarrow$) Suppose $f : X \to Y$ is a bijection. Define $g : Y \to X$ as follows. Given $y \in Y$, there exists $x \in X$ such that $f(x) = y$ since f is surjective. Moreover this element x is unique, since f is injective. So define $g(y) = x$ for the unique $x \in X$ for which $f(x) = y$. Then

- Given $x \in X$, we have $g(f(x))$ is the unique $a \in X$ such that $f(a) = f(x)$, so $g(f(x)) = x$.
- Given $y \in Y$, let $x \in X$ be such that $y = f(x)$. Then we have $f(g(y)) = f(g(f(x))) = f(x) = y$.

So $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and so g is an inverse for f.

($\Leftarrow$) Assume f has an inverse $g : Y \to X$. Then

- f is injective. Let $a, b \in X$ and assume that $f(a) = f(b)$. Then $a = g(f(a)) = g(f(b)) = b$.
- f is surjective. Let $c \in Y$ and define $a = g(c)$. Then $f(a) = f(g(c)) = c$.

So f is a bijection. $\square$

Example 7
Recall that the function $f : (0,1) \to (a,b)$ defined by $f(t) = a + t(b-a)$ is a bijection. Find an inverse for f.

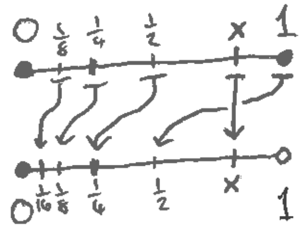
Define $g : (a,b) \longrightarrow (0,1)$ by $g(x) = \dfrac{x-a}{b-a}$ for all $x \in (a,b)$.

- g is well-defined: If $x \in (a,b)$, then $a < x < b$, so
$$0 = \frac{a-a}{b-a} < \frac{x-a}{b-a} < \frac{b-a}{b-a} = 1 \Rightarrow g(x) \in (0,1).$$
- $g \circ f = \mathrm{id}_{(0,1)}$. Let $t \in (0,1)$. Then
$$g(f(t)) = g(a + t(b-a)) = \frac{a + t(b-a) - a}{b-a} = t, \text{ as required.}$$
- $f \circ g = \mathrm{id}_{(a,b)}$. Let $x \in (a,b)$. Then
$$f(g(x)) = f\left(\frac{x-a}{b-a}\right) = a + \left(\frac{x-a}{b-a}\right)(b-a) = a + x - a = x$$
as required.

So g is an inverse for f. $\square$

Example 8
Find a bijection $f : [0,1] \to [0,1)$.

Idea of proof



We'll define $f : [0,1] \longrightarrow [0,1)$ so that $f(x) = x$ for most $x \in [0,1]$. But:

- We can't have $f(1) = 1$, so let $f(1) = \frac{1}{2}$
- We can't have $f(\frac{1}{2}) = \frac{1}{2}$, so let $f(\frac{1}{2}) = \frac{1}{4}$
- We can't have $f(\frac{1}{4}) = \frac{1}{4}$, so let $f(\frac{1}{4}) = \frac{1}{8}$
- ... and so on

So define $f(x) = \begin{cases} x & \text{if } x \neq 2^{-n} \text{ for any } n \in \mathbb{N} \\ 2^{-(n+1)} & \text{if } x = 2^{-n} \text{ for any } n \in \mathbb{N} \end{cases}$

We can (relatively) easily prove f is a bijection by defining an inverse.

Summary of proof strategies for *jections

Strategy (Proving a function is injective)
In order to prove that a function $f : X \to Y$ is injective, it suffices to fix $a, b \in X$, assume that $f(a) = f(b)$, and then derive $a = b$. ◁

Strategy (Proving a function is surjective)
To prove that a function $f : X \to Y$ is surjective, it suffices to take an arbitrary element $y \in Y$ and, in terms of y, find an element $x \in X$ such that $f(x) = y$.

In order to find such a value of x, it is often useful to start from the equation $f(x) = y$ and derive some possible values of x. But be careful—in order to complete the proof, it is necessary to verify that the equation $f(x) = y$ is true for the chosen value of x. ◁

Strategy (Proving a function is bijective)
To prove that a function $f : X \to Y$ is bijective, it suffices to either:

- Prove that f is injective, and that f is surjective; or
- Find an inverse $g : Y \to X$ for f, and verify that $g(f(x)) = x$ for all $x \in X$ and that $f(g(y)) = y$ for all $y \in Y$.

◁