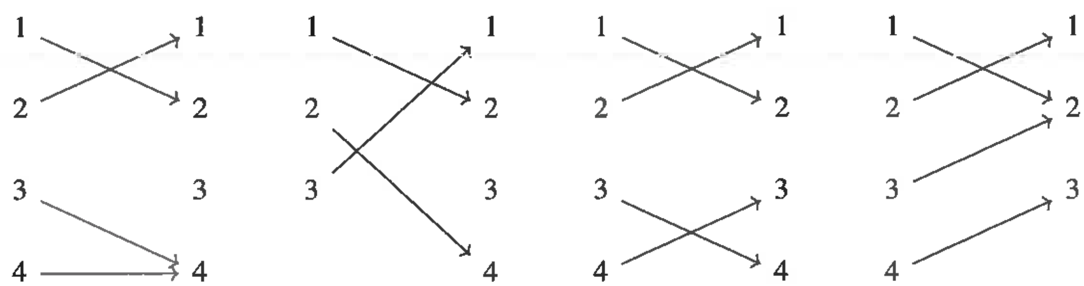
Math 300 Class 12

Monday 4th February 2019

Definition 1
A function $f : X \to Y$ is...

- ... **injective** if, for all $a, b \in X$, if $f(a) = f(b)$, then $a = b$;
- ... **surjective** if, for all $c \in Y$, there exists $a \in X$ such that $f(a) = c$;
- ... **bijective** if it is injective and surjective.

Example 2
For each of the following diagrams, determine whether the function it represents is: **(B)** bijective, **(I)** injective and not surjective, **(S)** surjective and not injective, or **(N)** neither injective nor surjective.



N	I	B	S

Example 3
Let $a, b \in \mathbb{R}$ with $a < b$. Find a bijection $(0,1) \to (a,b)$.

Define $f:(0,1) \longrightarrow (a,b)$ by $f(t) = a + t(b-a)$ for all $0 < t < 1$. [Note $\forall t \in (0,1)$, $a = a + 0(b-a) < a + t(b-a) < a + 1(b-a) = b$.]

- <u>f is injective.</u> Let $s, t \in (0,1)$ & assume $f(s) = f(t)$. Then
  $$a + s(b-a) = a + t(b-a) \overset{(-a)}{\Rightarrow} s(b-a) = t(b-a) \overset{(\div(b-a))}{\Rightarrow} s = t$$
  since $b - a \neq 0$
- <u>f is surjective.</u> Let $c \in (a,b)$ and define $t = \dfrac{c-a}{b-a}$.

Then $f(t) = a + \left(\dfrac{c-a}{b-a}\right)(b-a) = a + (c-a) = c$. $\square$

Example 4
Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that if f and g are injective, then $g \circ f$ is injective.
[Note: the same is true with 'injective' replaced by 'surjective' or 'bijective'.]

Assume f and g are injective.

Let $a, b \in X$ & assume $(g \circ f)(a) = (g \circ f)(b)$.

Then $g(f(a)) = g(f(b))$ (def. of $\circ$)

$\Rightarrow f(a) = f(b)$ (g is injective)

$\Rightarrow a = b$ (f is injective)

So $g \circ f$ is injective. $\square$

Definition 5
An **inverse** for a function $f : X \to Y$ is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Example 6
Find an inverse for the function you defined in Example 3.

Define $g : (a,b) \longrightarrow (0,1)$ by $g(x) = \frac{x-a}{b-a}$ for all $x \in (a,b)$.

- $\underline{g \circ f = \mathrm{id}_{(0,1)}}$ Let $t \in (0,1)$. Then

$$g(f(t)) = g(a + t(b-a)) = \frac{a + t(b-a) - a}{b-a} = t$$

- $\underline{f \circ g = \mathrm{id}_{(a,b)}}$ Let $x \in (a,b)$. Then

$$f(g(x)) = f\left(\frac{x-a}{b-a}\right) = a + \left(\frac{x-a}{b-a}\right)(b-a) = a + (x-a) = x.$$

So g is an inverse for f. $\square$

Theorem 7
A function $f : X \to Y$ is a bijection if and only if it has an inverse.

Proof
($\Rightarrow$) Suppose $f : X \to Y$ is a bijection. Define $g : Y \to X$ as follows. Given $y \in Y$, there exists $x \in X$ such that $f(x) = y$ since f is surjective. Moreover this element x is unique, since f is injective. So define $g(y) = x$ for the unique $x \in X$ for which $f(x) = y$. Then

- Given $x \in X$, we have $g(f(x))$ is the unique $a \in X$ such that $f(a) = f(x)$, so $g(f(x)) = x$.
- Given $y \in Y$, let $x \in X$ be such that $y = f(x)$. Then we have $f(g(y)) = f(g(f(x))) = f(x) = y$.

So $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and so g is an inverse for f.

($\Leftarrow$) Assume f has an inverse $g : Y \to X$. Then

- f is injective. Let $a, b \in X$ and assume that $f(a) = f(b)$. Then $a = g(f(a)) = g(f(b)) = b$.
- f is surjective. Let $c \in Y$ and define $a = g(c)$. Then $f(a) = f(g(c)) = c$.

So f is a bijection. $\square$

Summary of proof strategies for *jections

Strategy (Proving a function is injective)
In order to prove that a function $f : X \to Y$ is injective, it suffices to fix $a, b \in X$, assume that $f(a) = f(b)$, and then derive $a = b$. $\triangleleft$

Strategy (Proving a function is surjective)
To prove that a function $f : X \to Y$ is surjective, it suffices to take an arbitrary element $y \in Y$ and, in terms of y, find an element $x \in X$ such that $f(x) = y$.

In order to find such a value of x, it is often useful to start from the equation $f(x) = y$ and derive some possible values of x. But be careful—in order to complete the proof, it is necessary to verify that the equation $f(x) = y$ is true for the chosen value of x. $\triangleleft$

Strategy (Proving a function is bijective)
To prove that a function $f : X \to Y$ is bijective, it suffices to either:

- Prove that f is injective, and that f is surjective; or
- Find an inverse $g : Y \to X$ for f, and verify that $g(f(x)) = x$ for all $x \in X$ and that $f(g(y)) = y$ for all $y \in Y$.

$\triangleleft$