Math 300 Class 11

Friday 1st February 2019

Definition 1
Let $f : X \to Y$ be a function and let $U \subseteq X$. The **image of** U **under** f is the subset $f[U] \subseteq Y$ defined by

$$f[U] = \{f(x) \mid x \in U\} = \{y \in Y \mid \exists x \in U, y = f(x)\}$$

That is, $f[U]$ is the set of values that the function f takes when given inputs from U.

The '**image of** f' is the image of its entire domain, i.e. the set $f[X]$.

Example 2
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. Find $f[\mathbb{R}]$ and $f[(-1,1)]$.

$\boxed{f[\mathbb{R}] = [0,\infty)}$ Proof: Let $y \in \mathbb{R}$. Then $y \in f[\mathbb{R}]$
$\iff y = x^2$ for some $x \in \mathbb{R}$ $\iff y \geq 0 \iff y \in [0,\infty)$. □

$\boxed{f[(-1,1)] = [0,1)}$ Proof: Let $y \in \mathbb{R}$.
($\subseteq$) If $y \in f[(-1,1)]$, then $y = x^2$ for some $-1 < x < 1$, and so $0 \leq x^2 = y < 1$. So $y \in [0,1)$.
($\supseteq$) Let $y \in [0,1)$ and define $x = \sqrt{y} \in (-1,1)$. Then $f(x) = y$, so $y \in f[(-1,1)]$. □

Example 3
Define $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $g(a,b) = \dfrac{a}{1+|b|}$ for all $(a,b) \in \mathbb{R} \times \mathbb{R}$.

Find a subset $V \subseteq \mathbb{R} \times \mathbb{R}$ such that $g[V] = \mathbb{Q}$.

Define $V = \mathbb{Z} \times \mathbb{N} \subseteq \mathbb{R} \times \mathbb{R}$.

- ($\subseteq$) Let $x \in g[V]$. Then $x = g(a,b) = \frac{a}{1+|b|}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. But $1+|b| \in \mathbb{Z}$ and $1+|b| \neq 0$, so then $x \in \mathbb{Q}$.
- ($\supseteq$) Let $x \in \mathbb{Q}$. Then $x = \frac{a}{b}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \neq 0$. We may assume $b > 0$ — otherwise replace a and b with $-a$ and $-b$. But then $b-1 \in \mathbb{N}$ and $x = \frac{a}{b} = \frac{a}{1+(b-1)} = \frac{a}{1+|b-1|} = g(a, b-1)$

So $x \in g[\mathbb{Z} \times \mathbb{N}]$. □

Example 4
Let $f : X \to Y$ be a function and let $U, V \subseteq X$.

Prove that $f[U \cap V] \subseteq f[U] \cap f[V]$.

Let $y \in f[U \cap V]$. Then $y = f(x)$ for some $x \in U \cap V$.

So: • $x \in U$ (by def. of $\cap$), so $y \in f[U]$

• $x \in V$ (likewise) so $y \in f[V]$.

Hence $y \in f[U] \cap f[V]$, as required.

Give an example to show that $f[U] \cap f[V]$ need not be a subset of $f[U \cap V]$.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ for all $x \in \mathbb{R}$. Then $f[W] = \{1\}$ for all $\emptyset \neq W \subseteq \mathbb{R}$. So let $U = \{0\}$ and $V = \{1\}$. Then:

$f[U] \cap f[V] = \{1\} \cap \{1\} = \{1\}$ but $f[U \cap V] = f[\emptyset] = \emptyset$.

Definition 5
Let $f : X \to Y$ be a function and let $V \subseteq Y$. The **preimage of V under f** is the subset $f^{-1}[V] \subseteq X$ defined by

$$f^{-1}[V] = \{x \in X \mid f(x) \in V\} = \{x \in X \mid \exists y \in V, y = f(x)\}$$

Example 6
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. Find $f^{-1}[\mathbb{R}]$ and $f^{-1}[(-\infty, 4)]$.

$\boxed{f^{-1}[\mathbb{R}] = \mathbb{R}}$ Indeed, for all x, we have

$x \in f^{-1}[\mathbb{R}] \Leftrightarrow f(x) \in \mathbb{R}$, but this holds for all $x \in \mathbb{R}$, and so $f^{-1}[\mathbb{R}] = \mathbb{R}$.

$\boxed{f^{-1}[(-\infty, 4)] = (-2, 2)}$

($\subseteq$) Let $x \in f^{-1}[(-\infty, 4)]$. Then $x^2 = f(x) < 4$, so $-2 < x < 2$.

($\supseteq$) Let $x \in (-2, 2)$. Then $x^2 < 4$, so $x \in f^{-1}[(-\infty, 4)]$.

Example 7
Let $f : X \to Y$ be a function. Prove that $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$ for all subsets $U, V \subseteq Y$.

($\subseteq$) Let $x \in f^{-1}[U \cap V]$. Then $f(x) \in U \cap V$. So

$$\left.\begin{array}{lll} \cdot\ f(x) \in U & \Rightarrow & x \in f^{-1}[U] \\ \cdot\ f(x) \in V & \Rightarrow & x \in f^{-1}[V] \end{array}\right\} \Rightarrow x \in f^{-1}[U] \cap f^{-1}[V].$$

($\supseteq$) Let $x \in f^{-1}[U] \cap f^{-1}[V]$. Then

$$\left.\begin{array}{lll} \cdot\ x \in f^{-1}[U] & \Rightarrow & f(x) \in U \\ \cdot\ x \in f^{-1}[V] & \Rightarrow & f(x) \in V \end{array}\right\} \Rightarrow f(x) \in U \cap V$$

So $x \in f^{-1}[U \cap V]$. $\square$

Example 8
Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $W \subseteq Z$. Prove that $(g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]]$.

Let $x \in X$. Then

$$\begin{array}{lll} & x \in (g \circ f)^{-1}[W] & \\ \iff & (g \circ f)(x) \in W & \text{def of preimage} \\ \iff & g(f(x)) \in W & \text{def of } \circ \\ \iff & f(x) \in g^{-1}[W] & \text{def of preimage} \\ \iff & x \in f^{-1}[g^{-1}[W]] & \text{def of preimage} \end{array}$$

$\square$