Math 300 Class 10

Monday 28th January 2019

Definition 1 — *Functions*
A **function** f from a set X to a set Y is a specification of elements $f(x) \in Y$ for $x \in X$, such that

$$\forall x \in X,\ \exists! y \in Y,\ y = f(x)$$

We write $f : X \to Y$ to denote the assertion that f is a function **with domain X** and codomain Y.

Some terminology:

- Given $x \in X$, the (unique!) element $f(x) \in Y$ is called the **value** of f at x.
- The set X is called the **domain** (or **source**) of f;
- The set Y is called the **codomain** (or **target**) of f;

Some examples of functions (specifications are **well-defined**):

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 3x + 2$ for all $x \in \mathbb{R}$;
- $g : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $g(U) = X \setminus U$ for all $U \subseteq X$;
- $h : \mathbb{Z} \to \mathbb{N}$ defined by $h(n) = \begin{cases} 2n & \text{if } n \geqslant 0 \\ -(2n+1) & \text{if } n < 0 \end{cases}$;

Some non-examples of functions (specifications are not well-defined):

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ for all $x \in \mathbb{R}$;
- $g : \mathscr{P}(\mathbb{N}) \to \mathbb{N}$ defined by $g(U) =$ [the least element of U] for each $U \subseteq \mathbb{N}$;
- $h : \mathbb{Z} \to \mathbb{N}$ defined by $h(n) = \begin{cases} 2n & \text{if } n \geqslant 0 \\ -(2n+1) & \text{if } n \leqslant 0 \end{cases}$;

Axiom 2 — *Function extensionality*
Let $f : X \to Y$ and $g : A \to B$ be functions. Then $f = g$ if and only if f and g have the same domain and codomain, and $f(x) = g(x)$ for all $x \in X$.

Definition 3 — *Graph of a function*
Let $f : X \to Y$ be a function. The **graph** of f is the subset $\mathrm{Gr}(f) \subseteq X \times Y$ defined by

$$\mathrm{Gr}(f) = \{(x, f(x)) \mid x \in X\} = \{(x,y) \in X \times Y \mid y = f(x)\}$$

Example 4
Find a function $f : \mathbb{N} \to \mathbb{N}$ whose graph equal to the set $\{(a,b) \in \mathbb{N} \times \mathbb{N} \mid 4a+1 = 2b-1\}$.

Define $f(n) = 2n+1$ for all $n \in \mathbb{N}$. Then:

$$\begin{aligned}
(a,b) \in \mathrm{Gr}(f) &\Leftrightarrow b = f(a) \\
&\Leftrightarrow b = 2a+1 \\
&\Leftrightarrow 2b = 4a+2 \\
&\Leftrightarrow 2b-1 = 4a+1 \\
&\Leftrightarrow (a,b) \in \text{the above set}
\end{aligned}$$

Example 5
Let $f, g : X \to Y$. Prove that if $\mathrm{Gr}(f) = \mathrm{Gr}(g)$, then $f = g$.

Suppose $\mathrm{Gr}(f) = \mathrm{Gr}(g)$ and let $x \in X$.

Then $(x, f(x)) \in \mathrm{Gr}(f)$ by def. of graph

$\Rightarrow (x, \cancel{g}f(x)) \in \mathrm{Gr}(g)$ since $\mathrm{Gr}(f) = \mathrm{Gr}(g)$

$\Rightarrow (x, f(x)) = (x, g(x))$ since $g(x)$ is the unique $y \in Y$ such that $(x,y) \in \mathrm{Gr}(g)$

$\Rightarrow f(x) = g(x)$ by def of ordered pairs.

So $f = g$ by function extensionality.

Definition 6 — *Identity function*
The **identity function** on a set X is the function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(a) = a$ for all $a \in X$.

Example 7
Describe the set $\mathrm{Gr}(\mathrm{id}_X)$.

$$\begin{aligned}
\mathrm{Gr}(\mathrm{id}_X) &= \{(x, \mathrm{id}_X(x)) \mid x \in X\} \\
&= \{(x,x) \mid x \in X\}
\end{aligned}$$

[This set is called the <u>diagonal</u> <u>subset</u> of X, and is also denoted by Δ_X.]

Definition 8 — *Composition of functions*
Given functions $f : X \to Y$ and $g : Y \to Z$, the **composite** of f and g is the function $g \circ f : X \to Z$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in X$.

Example 9 — *Associativity of composition*
Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ be functions. Prove that $h \circ (g \circ f) = (h \circ g) \circ f : X \to W$.

Let $x \in X$. Then

$$
\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) && \text{def. of } \circ \\
&= h(g(f(x))) && \text{def. of } \circ \\
&= (h \circ g)(f(x)) && \text{def. of } \circ \\
&= ((h \circ g) \circ f)(x) && \text{def. of } \circ
\end{aligned}
$$

So $h \circ (g \circ f) = (h \circ g) \circ f$ by function extensionality.

Example 10
Let $f : X \to Y$ and $g, h : Y \to X$. Prove that if $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$, then $g = h$.

Let $y \in Y$. If $g \circ f = \mathrm{id}_X$ & $f \circ h = \mathrm{id}_Y$, then

$$
\begin{aligned}
g(y) &= g(\mathrm{id}_Y(y)) && \text{by def of } \mathrm{id}_Y \\
&= g((f \circ h)(y)) && \text{by } \cancel{\text{def of}} \text{ assumption} \\
&= (g \circ (f \circ h))(y) && \text{by def of } \circ \\
&= ((g \circ f) \circ h)(y) && \text{by Ex 9} \\
&= (g \circ f)(h(y)) && \text{by def of } \circ \\
&= \mathrm{id}_X(h(y)) && \text{by assumption} \\
&= h(y) && \text{by def of } \mathrm{id}_X
\end{aligned}
$$

So $g = h$ by fn extensionality.

Definition 11 — *Sequences of real numbers*
A **sequence of real numbers** is a function $x : \mathbb{N} \to \mathbb{R}$. Given a sequence x, we write x_n instead of $x(n)$ and write $(x_n)_{n \geqslant 0}$, or even just (x_n), instead of $x : \mathbb{N} \to \mathbb{R}$.

The values x_n are called the **terms** of the sequence, and the variable n is called the **index** of the term x_n. Examples of sequences include:

- (x_n), defined by $x_n = \frac{n}{n+1}$ for all $n \in \mathbb{N}$;
- (y_n), defined by $y_n = 2^n$ for all $n \in \mathbb{N}$.

Definition 12 — *Convergence of a sequence*
Let (x_n) be a sequence and let $a \in \mathbb{R}$. We say that (x_n) **converges to** a, and write $(x_n) \to a$ (LaTeX code: `\to`), if the following condition holds:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, x_n \in (a - \varepsilon, a + \varepsilon)$$

The value a is called a **limit** of (x_n).

Moreover, we say that a sequence (x_n) **converges** if it has a limit, and diverges otherwise.

Example 13
Prove that the sequence (x_n) defined above converges to 1.

Let $\varepsilon > 0$, and define N to be some natural number $> -1 + \frac{1}{\varepsilon}$.
If $n \geqslant N$ then $n > -1 + \frac{1}{\varepsilon} \Rightarrow n + 1 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n+1} < \varepsilon$.
So $1 - \varepsilon < 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = x_n < 1 < 1 + \varepsilon$
So $x_n \in (1-\varepsilon, 1+\varepsilon)$, as required.

Example 14
Prove that the sequence (y_n) defined above diverges.

$\forall a \in \mathbb{R}, \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geqslant N, y_n \notin (a-\varepsilon, a+\varepsilon$

Let $a \in \mathbb{R}$, and define $\varepsilon = 1$. Let $N \in \mathbb{N}$.
Take $n \geqslant N$ such that $a < 2^n - 1$ — such a value exists since 2^n is unbounded.
Then $y_n = 2^n > a + 1 \Rightarrow y_n \notin (a-1, a+1)$
So (y_n) diverges.

Ex 13 scratch work:

$$\frac{n}{n+1} \in (1-\varepsilon, 1+\varepsilon)$$

$$\iff \quad 1-\varepsilon < \frac{n}{n+1} < 1+\varepsilon$$

$$\iff \quad -\varepsilon < \frac{n}{n+1} - 1 < \varepsilon$$

Now $\displaystyle \frac{n}{n+1} - 1 = \frac{n-(n+1)}{n+1} = \frac{-1}{n+1} < 0$

So we need $\displaystyle -\varepsilon < \frac{-1}{n+1}$

$$\iff \quad \varepsilon > \frac{1}{n+1}$$

$$\iff \quad \frac{1}{\varepsilon} < n+1$$

$$\iff \quad \underline{\underline{n > -1 + \frac{1}{\varepsilon}}}$$