Math 300 Class 7

Friday 18th January 2019

So far the only tool we have at our disposal for proving that a proposition is *false* is to assume that it is true and derive a contradiction. In this class we'll use logical equivalence to derive other means of proving when something is false.

Definition 1 — *Maximally negated logical formulae*
A logical formula is **maximally negated** if the only instances of the negation operator $\neg$ appear immediately before a predicate (other proposition involving no logical operators or quantifiers).

Example 2
Identify which of the following logical formulae are maximally negated.

$$[p \wedge (q \Rightarrow (\neg r))] \Leftrightarrow (s \wedge (\neg t)) \quad ✓$$

$$(\neg\neg q) \Rightarrow q \quad ✗$$

$$(\neg p(x)) \Rightarrow \forall y \in X, \neg(r(x,y) \wedge s(x,y)) \quad ✗$$

$$(\neg p(x)) \Rightarrow \forall y \in X, (\neg r(x,y)) \vee (\neg s(x,y)) \quad ✓$$

$$\forall x \in \mathbb{R}, [x > 1 \Rightarrow (\exists y \in \mathbb{R}, [x < y \wedge \neg(x^2 \leqslant y)])] \quad ✓$$

$$\neg \exists x \in \mathbb{R}, [x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leqslant y])] \quad ✗$$

Theorem 3
Every logical formula (built using only the logical operators and quantifiers we have seen so far) is logically equivalent to a maximally negated logical formula. □

The precise proof of Theorem 3 is not yet in our reach, but we can derive an algorithm for maximally negating a logical formula by working out how to maximally negate each logical operator and quantifier.

Theorem 4 — *Law of double negation*
Let p be a propositional variable. Then $p \equiv \neg\neg p$.

Proof

By truth table:

p	$\neg p$	$\neg\neg p$
✓	✗	✓
✗	✓	✗

The columns for p and $\neg\neg p$ are identical.

□

Theorem 5 — *de Morgan's laws for logical operators*
Let p and q be logical formulae. Then:
(a) $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$; and
(b) $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

Proof

By truth table again:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$	$(\neg p) \vee (\neg q)$
✓	✓	✓	✗	✓	✗	✗	✗	✗	✗
✓	✗	✗	✓	✓	✗	✗	✓	✗	✓
✗	✓	✗	✓	✓	✗	✓	✗	✗	✓
✗	✗	✗	✓	✗	✓	✓	✓	✓	✓

identical

identical

□

Theorem 6
Let p and q be logical formulae. Then $\neg(p \Rightarrow q) \equiv p \wedge (\neg q)$.

Proof

By ... truth table!

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge (\neg q)$
✓	✓	✓	✗	✗	✗
✓	✗	✗	✓	✓	✓
✗	✓	✓	✗	✗	✗
✗	✗	✓	✗	✓	✗

□

identical

Theorem 7 — *de Morgan's laws for quantifiers*
let $p(x)$ be a logical formula with free variable x ranging over a set X. Then:

(a) $\neg\forall x \in X, p(x) \equiv \exists x \in X, \neg p(x)$; and

(b) $\neg\exists x \in X, p(x) \equiv \forall x \in X, \neg p(x)$.

Proof of (b).

($\Rightarrow$) Assume $\neg\exists x \in X, p(x)$.
To prove $\forall x \in X, \neg p(x)$, we let $x \in X$ be arbitrary, assume $p(x)$, and derive a contradiction.
So fix $x \in X$ and assume $p(x)$. Then $\exists x \in X, p(x)$ is true — this contradicts our assumption!
So $\neg p(x)$ is true, and so $\forall x \in X, \neg p(x)$ is true.

($\Leftarrow$) Assume $\forall x \in X, \neg p(x)$.
To prove $\neg\exists x \in X, p(x)$, we assume $\exists x \in X, p(x)$ and derive a contradiction.
So assume $\exists x \in X, p(x)$, and let $a \in X$ be an element such that $p(a)$ is true.
Then $\neg p(a)$ is true too, since we're assuming $\forall x \in X, \neg p(x)$. This is a contradiction.
So $\neg\exists x \in X, p(x)$ is true.

□

Part (a) of Theorem 7 is so important that the proof strategy it suggests has a name.

Strategy (Proof by counterexample)
To prove that a proposition of the form $\forall x \in X, p(x)$ is false, it suffices to find a single element $a \in X$ such that $p(a)$ is false. The element a is called a **counterexample** to the proposition. ◁

Piecing this all together, we obtain the following, which summarises everything we just proved:

Negation outside		Negation inside	Proof
$\neg(p \wedge q)$	$\equiv$	$(\neg p) \vee (\neg q)$	Theorem 5(a)
$\neg(p \vee q)$	$\equiv$	$(\neg p) \wedge (\neg q)$	Theorem 5(b)
$\neg(p \Rightarrow q)$	$\equiv$	$p \wedge (\neg q)$	Theorem 6
$\neg(\neg p)$	$\equiv$	p	Theorem 4
$\neg\forall x \in X, p(x)$	$\equiv$	$\exists x \in X, \neg p(x)$	Theorem 7(a)
$\neg\exists x \in X, p(x)$	$\equiv$	$\forall x \in X, \neg p(x)$	Theorem 7(b)

We can use these equivalences to maximally negate logical formulae by iteratively pushing the negation operator inside the logical formula.

Example 8

Find a maximally negated propositional formula that is logically equivalent to $\neg(p \Leftrightarrow q)$. [It might help you to recall that $p \Leftrightarrow q$ is defined to mean $(p \Rightarrow q) \wedge (q \Rightarrow p)$.]

$$
\begin{aligned}
\neg(p \Leftrightarrow q) &\equiv \neg((p \Rightarrow q) \wedge (q \Rightarrow p)) && \text{by def of } \Leftrightarrow \\
&\equiv [\neg(p \Rightarrow q)] \vee [\neg(q \Rightarrow p)] && \text{by dM for } \wedge \\
&\equiv (p \wedge (\neg q)) \vee (q \wedge (\neg p)) && \text{negating } \Rightarrow
\end{aligned}
$$

What strategy does this equivalecne suggest for proving that a proposition of the form $p \Leftrightarrow q$ is false?

To prove $p \Leftrightarrow q$ is false, it suffices to either

(i) Prove p is true and q is false; or

(ii) Prove q is true and p is false.

Example 9

Maximally negate the following logical formula, then prove that it is true or prove that it is false.

$$\exists x \in \mathbb{R}, [x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leqslant y])]$$

$$\begin{aligned} & \neg \exists x \in \mathbb{R}, [x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leq y])] \\ \equiv\ & \forall x \in \mathbb{R}, \neg[x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leq y])] && \neg\exists \\ \equiv\ & \forall x \in \mathbb{R}, [x \leq 1 \vee \neg(\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leq y])] && \neg\wedge \\ \equiv\ & \forall x \in \mathbb{R}, [x \leq 1 \vee (\exists y \in \mathbb{R}, \neg[x < y \Rightarrow x^2 \leq y])] && \neg\forall \\ \equiv\ & \forall x \in \mathbb{R}, [x \leq 1 \vee (\exists y \in \mathbb{R}, [x < y \wedge x^2 > y])] && \neg\Rightarrow \end{aligned}$$

The statement is false. To prove its negation, we'll use the fact that $p \vee q \equiv (\neg p) \Rightarrow q$ to prove:

$$\forall x \in \mathbb{R}, [x > 1 \Rightarrow (\exists y \in \mathbb{R}, [x < y \wedge x^2 > y])]$$

So fix $x \in \mathbb{R}$ and assume $x > 1$. Then $x^2 > x$, so $x^2 - x > 0$.

Define $y = \dfrac{x + x^2}{2} = x + \dfrac{(x^2 - x)}{2} = x^2 - \dfrac{(x^2 - x)}{2}$

Then $x < y$ since $x^2 - x > 0$ and $y = x + \frac{(x^2 - x)}{2}$

and $x^2 > y$ since $x^2 - x > 0$ and $y = x^2 - \frac{(x^2 - x)}{2}$

as required.

$\square$