Math 300 Class 5

Monday 14th January 2019

Definition 1
Let p and q be logical formulae. We say that p and q are **logically equivalent**, and write $p \equiv q$, if q can be derived from p and p can be derived from q.

Example 2
The propositional formula $p \vee q$ is equivalent to $(\neg p) \Rightarrow q$.

($\Rightarrow$) Assume $p \vee q$. To prove $(\neg p) \Rightarrow q$, assume $\neg p$ is true.

We need to show q is true.

Since $p \vee q$ is true, either p or q is true.

- If p is true, we have a contradiction $\because$ $\neg p$ is true.
- If q is true, then there is nothing left to prove

$\Rightarrow$ q is true, as required.

($\Leftarrow$) Assume $(\neg p) \Rightarrow q$. We need to derive $p \vee q$.

By the law of excluded middle, $p \vee \neg p$ is true.

- If p is true, then $p \vee q$ is true by def. of $\vee$.
- If $\neg p$ is true, then q is true since $(\neg p) \Rightarrow q$. But then $p \vee q$ is true by def. of $\vee$.

In both cases, $p \vee q$ is true, as required.

Logical equivalence allows us to cook up new proof strategies. For instance, Example 2 suggests:

Strategy
In order to prove a proposition of the form $p \vee q$, it suffices to assume that p is false and derive that q is true. ◁

Example 3
Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, and assume that $x^2 = n$. Then either $\underbrace{x \text{ is irrational}}_{p}$, $\underbrace{\text{or}}_{\vee}$ $\underbrace{x \text{ is an integer}}_{q}$. $\equiv (\neg p) \Rightarrow q$

Proof.

Assume $x \in \mathbb{Q}$.

Then $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$.

Since $x^2 = n$, we have $\frac{a^2}{b^2} = n \Rightarrow a^2 = nb^2$.

All prime factors of a^2 & b^2 appear raised to even powers

$\Rightarrow$ all prime factors of n appear raised to even powers

$\Rightarrow$ n is a perfect square

We're secretly using the fundamental thm of arithmetic here — this is Thm 4.2.12 in the book.

$\Rightarrow$ $x = \pm\sqrt{n} \in \mathbb{Z}$, as required. □

Truth tables

Definition 4
The **truth table** of a propositional formula is the table with one row for each possible assignment of truth values to its constituent propositional variables, which displays the truth values of the propositional formula (and possibly its subformulae).

Example 5
The following are the truth tables for $\neg p$, $p \wedge q$, $p \vee q$ and $p \Rightarrow q$.

p	$\neg p$
✓	×
×	✓

p	q	$p \wedge q$
✓	✓	✓
✓	×	×
×	✓	×
×	×	×

p	q	$p \vee q$
✓	✓	✓
✓	×	✓
×	✓	✓
×	×	×

p	q	$p \Rightarrow q$
✓	✓	✓
✓	×	×
×	✓	✓
×	×	✓

"Principle of explosion" — see Axiom 1.1.49 in the book.

Theorem 6
Two propositional formulae are logically equivalent if and only if their truth values are the same under any assignment of truth values to their constituent propositional variables.

In light of Theorem 6, we can prove that two propositional formulae are logically equivalent by showing that their columns in a truth table are identical.

Example 7
By filling in the columns of the following truth table, prove that the formulae $p \vee q$ and $(\neg p) \Rightarrow q$ are logically equivalent.

p	q	$p \vee q$	$\neg p$	$(\neg p) \Rightarrow q$	
✓	✓	✓	×	✓	× ⇒ ✓
✓	×	✓	×	✓	× ⇒ ×
×	✓	✓	✓	✓	✓ ⇒ ✓
×	×	×	✓	×	✓ ⇒ ×

The columns for $p \vee q$ and $(\neg p) \Rightarrow q$ are identical, so $p \vee q \equiv (\neg p) \Rightarrow q$.

Example 8
By filling in the columns of the following truth table, prove that the formulae $p \Rightarrow q$ and $(\neg q) \Rightarrow (\neg p)$ are logically equivalent.

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$(\neg q) \Rightarrow (\neg p)$
✓	✓	✓	×	×	✓
✓	×	×	✓	×	×
×	✓	✓	×	✓	✓
×	×	✓	✓	✓	✓

The columns for $p \Rightarrow q$ and $(\neg q) \Rightarrow (\neg p)$ are identical, so $p \Rightarrow q \equiv (\neg q) \Rightarrow (\neg p)$.

The proposition $(\neg q) \Rightarrow (\neg p)$ is called the **contrapositive** of $p \Rightarrow q$.

Strategy (Proof by contraposition)
In order to prove a proposition of the form $p \Rightarrow q$, it suffices to assume that q is false and derive that p is false. ◁

Example 9

Prove that for all $n \in \mathbb{Z}$, if $\underbrace{n^2 \text{ is even}}_{p}$, then $\underbrace{n \text{ is even}}_{q}$.

$p \Rightarrow q \equiv (\neg q) \Rightarrow (\neg p)$

Assume n is odd.

Then $n = 2k+1$ for some $k \in \mathbb{Z}$

$\Rightarrow n^2 = 4k^2 + 4k + 1 = 2\underbrace{(2k^2+2k)}_{\in \mathbb{Z}} + 1$

$\Rightarrow n^2$ is odd. So if n^2 is even, then n is even. $\square$

Example 10

Construct a truth table to prove that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
✓	✓	✓	✓	✓	✓	✓	✓
✓	✓	✗	✓	✓	✓	✗	✓
✓	✗	✓	✓	✓	✗	✓	✓
✓	✗	✗	✗	✗	✗	✗	✗
✗	✓	✓	✓	✗	✗	✗	✗
✗	✓	✗	✓	✗	✗	✗	✗
✗	✗	✓	✓	✗	✗	✗	✗
✗	✗	✗	✗	✗	✗	✗	✗

The columns for $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are identical, so $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

Pre-class assignment for Class 6 (Wed, Jan 16)

On Canvas, go to Pages → LaTeX resources → Getting started and follow the instructions to either set up a (free) online account with Overleaf or install a LaTeX implementation on your computer. Download the LaTeX template file (`template.tex`). Use your LaTeX editor to change the name and date to your own name and the current date, and then compile it as a PDF document. Finally, upload the `.tex` file on Canvas (go to Assignments → Class 6).