

Math 300 Class 3

Wednesday 9th January 2019

Propositional formulae and logical operators

Definition 1

A **propositional formula** is an expression that is built out of **propositional variables** p, q, r, \dots using **logical operators** (to be defined soon).

The main logical operators of interest to us are:

- **Conjunction** (\wedge), where $p \wedge q$ represents 'p and q';
- **Disjunction** (\vee), where $p \vee q$ represents 'p or q';
- **Implication** (\Rightarrow), where $p \Rightarrow q$ represents 'if p, then q';
- **Negation** (\neg), where $\neg p$ represents 'not p'.

A fifth logical operator, *biimplication* (\Leftrightarrow), is defined in terms of \Rightarrow and \wedge by letting $p \Leftrightarrow q$ be shorthand for $(p \Rightarrow q) \wedge (q \Rightarrow p)$; we read $p \Leftrightarrow q$ as 'p if and only if q'.

Exercise 2

Fix integers a and b . Let p represent ' $a + b$ is even', let q represent ' a is even', and let r represent ' b is odd'. Consider the following logical formula:

$$p \Leftrightarrow ((q \wedge r) \vee ((\neg q) \wedge (\neg r)))$$

Translate it into plain English.

" $a + b$ is even if and only if a and b are both even, or a and b are both odd"

The precise definitions of \wedge , \vee , \Rightarrow and \neg are *important* (and in the book), but the most useful thing for us is the *proof strategies* that they suggest. In this class we focus on a handful of examples—a more complete list is on page 5

Proving implications

You will often find that a goal in your proof is of the form $p \Rightarrow q$. In this case, the goal can be proved by *assuming* p and *deriving* q .

Exercise 3

Let $m, n \in \mathbb{N}$. Prove that if $2m + 1 = 2n + 1$, then $m = n$.

Suppose that $2m + 1 = 2n + 1$.

Then $2m = 2n$ (by subtracting 1)

So $m = n$ (by dividing by 2)

as required. \square

Proving biimplications

The biimplication $p \Leftrightarrow q$ asserts that p and q are somehow *equivalent*. A typical proof consists of two steps: first prove $p \Rightarrow q$, and then prove $q \Rightarrow p$.

Exercise 4

Let $x \in \mathbb{R}$. Prove that $x \in \mathbb{Q}$ if and only if $kx \in \mathbb{Z}$ for some ^{nonzero} $k \in \mathbb{Z}$.

(\Rightarrow) Suppose $x \in \mathbb{Q}$. Then $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Hence $bx = a \in \mathbb{Z}$, as required (take $k = b$).

(\Leftarrow) Suppose $kx \in \mathbb{Z}$ for some nonzero $k \in \mathbb{Z}$.

Then $x = \frac{kx}{k}$ is the quotient of an integer by a nonzero integer. So $x \in \mathbb{Q}$, as required. \square

Proof by cases

When an *assumption* in a proof takes the form $p \vee q$, this means that we know that at least one of p or q is true, but we don't necessarily know which. As such, we need to split into cases based on whether p is true or q is true, and complete the proof in both cases.

Exercise 5

Prove that every positive proper divisor of 9 is odd.

Suppose n is a positive proper divisor of 9. Then either $n=1$ or $n=3$.

• If $n=1$, then n is odd $\because n = 2 \times 0 + 1$

• If $n=3$, then n is odd $\because n = 2 \times 1 + 1$

In both cases, we see that n is odd. \square

A useful example of when you might use proof by cases is the *law of excluded middle*:

Axiom 6 — *Law of excluded middle*

Let p be any proposition. Then $p \vee (\neg p)$ is true.

Exercise 7

Let x and y be irrational numbers. Prove that x^y may be rational.

Either $\sqrt{2}^{\sqrt{2}}$ is rational, or it is not.

• If $\sqrt{2}^{\sqrt{2}}$ is rational, then let $x = \sqrt{2}$ and $y = \sqrt{2}$.

Then x and y are irrational, and x^y is rational.

• If $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$.

Then x and y are irrational, and

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$$

So x^y is rational.

In both cases, we see that x^y is rational for some irrational x and y . \square

Proof by contradiction

We can only prove that things are *true*, but saying that p is false is exactly the same as saying that $\neg p$ is true. Proofs of negations are called proofs *by contradiction*.

Definition 8

A **contradiction** is a proposition that is known or assumed to be false.

In order to prove a goal of the form $\neg p$, we assume p is true and derive a contradiction.

Exercise 9

Let $x \in \mathbb{Q}$. Prove that $\sqrt{2} + x$ is irrational.

Towards a contradiction, assume $\sqrt{2} + x$ is rational.

Then:

- $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$
 $\therefore x$ is rational

- $\sqrt{2} + x = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$
by assumption

$$\begin{aligned}\Rightarrow \sqrt{2} &= (\sqrt{2} + x) - x \\ &= \frac{c}{d} - \frac{a}{b} \\ &= \frac{cb - ad}{bd} \quad \Rightarrow \sqrt{2} \in \mathbb{Q}\end{aligned}$$

This is a contradiction. So $\sqrt{2} + x$ is irrational. \square

Pre-class assignment for Class 4 (Fri, Jan 11)

Read §1.2 *Variables and quantification* up to (but excluding) Definition 1.2.9, and then answer the questions on Canvas (go to Assignments \rightarrow Class 4).

You do not need to complete Exercises 1.2.6 and 1.2.8.

Strategies for proving propositional formulae

Strategy (Proving conjunctions)

A proof of the proposition $p \wedge q$ can be obtained by tying together two proofs, one being a proof that p is true and one being a proof that q is true. ◀

Strategy (Proving disjunctions)

In order to prove a proposition of the form $p \vee q$, it suffices to prove just one of p or q . ◀

Strategy (Proving implications)

In order to prove a proposition of the form $p \Rightarrow q$, it suffices to assume that p is true, and then derive q from that assumption. ◀

Strategy (Proving negations—proof by contradiction)

In order to prove a proposition p is false (that is, that $\neg p$ is true), it suffices to assume that p is true and derive a contradiction. ◀

Strategies for using propositional formulae as assumptions

Strategy (Assuming disjunctions)

If an assumption in a proof has the form $p \wedge q$, then we may assume p and assume q in the proof. ◀

Strategy (Assuming disjunctions—proof by cases)

If an assumption in a proof has the form $p \vee q$, then we may derive a proposition r by splitting into two cases: first, derive r from the temporary assumption that p is true, and then derive r from the temporary assumption that q is true. ◀

Strategy (Assuming implications—modus ponens)

If an assumption in a proof has the form $p \Rightarrow q$, and p is also assumed to be true, then we may also assume that q is true. ◀

Strategy (Assuming negations)

If an assumption in a proof has the form $\neg p$, then any derivation of p leads to a contradiction. ◀

The law of excluded middle

Strategy (Using the law of excluded middle)

In order to prove a proposition q is true, it suffices to split into cases based on whether some other proposition p is true or false, and prove that q is true in each case. ◀