1. [*Colley, §4.3 Q36*] Find the maximum value of $\sin\alpha\sin\beta\sin\gamma$, where α, β and γ are the interior angles of a triangle.

We need to maximise $\underbrace{\sin\alpha\sin\beta\sin\gamma}_{f(\alpha,\beta,\gamma)}$ subject to $\begin{cases} \alpha, \beta, \gamma > 0 \ \& < \pi \\ \underbrace{\alpha+\beta+\gamma}_{g(\alpha,\beta,\gamma)} = \pi \end{cases}$ constraint

$$\nabla f = \lambda \nabla g \iff \begin{pmatrix} \cos\alpha \sin\beta \sin\gamma \\ \sin\alpha \cos\beta \sin\gamma \\ \sin\alpha \sin\beta \cos\gamma \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{array}{l} \leftarrow (1) \\ \leftarrow (2) \\ \leftarrow (3) \end{array}$$

(1) & (2) $\Rightarrow$ $\cos\alpha\sin\beta\sin\gamma = \sin\alpha\cos\beta\sin\gamma$

$\Rightarrow (\cos\alpha\sin\beta - \sin\alpha\cos\beta)\sin\gamma = 0$

$\Rightarrow \sin(\beta-\alpha)\sin\gamma = 0$

(1) & (3) $\Rightarrow$ $\cos\alpha\sin\beta\sin\gamma = \sin\alpha\sin\beta\cos\gamma$

$\Rightarrow (\cos\alpha\sin\gamma - \sin\alpha\cos\gamma)\sin\beta = 0$

$\Rightarrow \sin(\gamma-\alpha)\sin\beta = 0$

If $\sin\gamma = 0$ or $\sin\beta = 0$ then $\sin\alpha\sin\beta\sin\gamma = 0$

Otherwise, $\sin(\beta-\alpha) = 0$ & $\sin(\gamma-\alpha) = 0$

$\Rightarrow \beta - \alpha = 0$ and $\gamma - \alpha = 0$ $(\because 0 < \alpha, \beta, \gamma < \pi)$

$\Rightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$ $(\because \alpha+\beta+\gamma = \pi)$

$\Rightarrow \sin\alpha\sin\beta\sin\gamma = \left(\sin\frac{\pi}{3}\right)^3 = \left(\frac{\sqrt{3}}{2}\right)^3 = \underline{\underline{\frac{3\sqrt{3}}{8}}} > 0$

maximum

2. Let Ω be the plane in $\mathbb{R}^3$ described by the equation $2x - z = 2$. Find the point P on Ω that is closest to the point $Q(4, -1, 1)$...

* (a) ...by using Lagrange multipliers;

(b) ...using least squares approximation;

(c) ...by reasoning geometrically.

We want to minimise $\underbrace{(x-4)^2 + (y+1)^2 + (z-1)^2}_{f}$

subject to $\underbrace{2x - z}_{g} = 2$

$$\nabla f = \lambda \nabla g \iff \begin{pmatrix} 2(x-4) \\ 2(y+1) \\ 2(z-1) \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

This + the constraint yield the following linear system:

$$\begin{cases} 2x \quad -z \quad = 2 & -① \\ 2x \quad -2\lambda = 8 & -② \\ 2y = -2 & -③ \\ 2z + \lambda = 2 & -④ \end{cases}$$

③ $- \boxed{y = -1}$

①−②: $-z + 2\lambda = -6$

④: $2z + \lambda = 2$

$\Rightarrow 5\lambda = -10$

$\Rightarrow \lambda = -2$

$\Rightarrow 2x + 4 = 8 \Rightarrow \boxed{x = 2}$ & $2z - 2 = 2 \Rightarrow \boxed{z = 2}$

So the minimum value of $(x-4)^2 + (y+1)^2 + (z-1)^2$ on the plane is $(-2)^2 + 0^2 + 1^2 = 5$

(Not a max, e.g. $(1,0,0)$ is on the plane but $(-3)^2 + 1^2 + 0^2 = 10 > 5$.)

So the pt on the plane closest to Q is $\underline{\underline{(2, -1, 2)}}$.

2. Let Ω be the plane in $\mathbb{R}^3$ described by the equation $2x - z = 2$. Find the point P on Ω that is closest to the point $Q(4, -1, 1)$...

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Normal vector : $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ $\Rightarrow$ a basis for parallel plane through $\vec{0}$ is given by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow$ each pt on Ω is of the form $\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{pt on plane}} + s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

directions // to plane

We find the least squares solution to

$$\begin{cases} 1 + s = 4 \\ t = -1 \\ 2s = 1 \end{cases} \quad \text{i.e.} \quad \begin{cases} s = 3 \\ t = -1 \\ 2s = 1 \end{cases}$$

Note: columns are LI so A^TA is invertible

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \Rightarrow A^TA = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (A^TA)^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow A^T\vec{b} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} s^* \\ t^* \end{pmatrix} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{5}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow s^* = 1, \quad t^* = -1$$

So the pt on the plane closest to $(4, -1, 1)$ is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

2. Let Ω be the plane in $\mathbb{R}^3$ described by the equation $2x - z = 2$. Find the point P on Ω that is closest to the point $Q(4, -1, 1)$...

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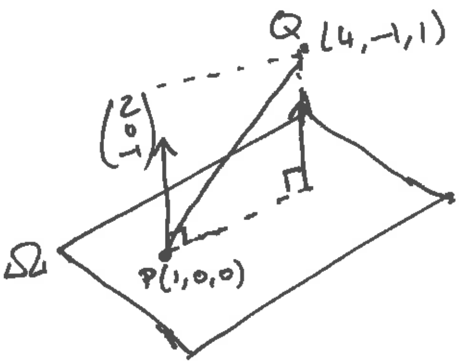


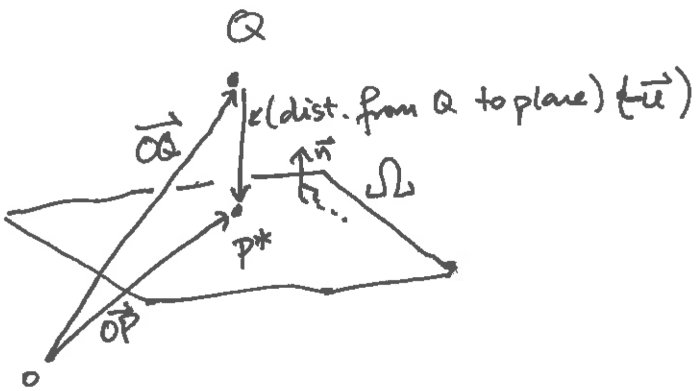
Distance from Q to plane:

pt on plane: $P(1,0,0)$

normal vector to plane: $\vec{n} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$$\Rightarrow \text{distance} = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|} = \frac{\left|\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}\right|}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Unit vector in direction of $\vec{n}$: $\vec{u} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$



$\Rightarrow$ if P^* is the pt on the plane closest to Q, it is $\sqrt{5}$ units from Q in the direction of $-\vec{u}$

$$\Rightarrow \vec{OP^*} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} - \sqrt{5}\,\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

3. Find a 2×2 symmetric matrix A of greatest determinant such that $\left\|A\begin{pmatrix}1\\1\end{pmatrix}\right\| = 2$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ is an eigenvector of A with eigenvalue 1.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$

so $\left\|A\begin{pmatrix}1\\1\end{pmatrix}\right\| = 2 \iff \left\|A\begin{pmatrix}1\\1\end{pmatrix}\right\|^2 = 4 \iff (a+b)^2 + (c+d)^2 = 4$

$A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \iff \begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \iff \begin{cases} a-b=1 \\ c-d=-1 \end{cases}$

At this point, we <u>could</u> solve the follwing constraint problem:

Maximise $ad-bc$ subject to $\begin{cases} (a+b)^2 + (c+d)^2 = 4 \\ a-b=1 \\ c-d=-1 \end{cases}$

Since b & c are so easy to isolate from the 2^{nd} & 3^{rd} constraints, we'll substitute instead:

$A = \begin{pmatrix} a & a-1 \\ d-1 & d \end{pmatrix}$, $(a+b)^2 + (c+d)^2 = (2a-1)^2 + (2d-1)^2 = 4$

So we'll maximise $\underbrace{ad - (a-1)(d-1)}_{f(a,d)}$ $= a+d-1$ subject to $\underbrace{(2a-1)^2 + (2d-1)^2 = 4}_{g(a,d)=k}$

$\nabla f = \lambda \nabla g$: $\begin{pmatrix}1\\1\end{pmatrix} = \lambda\begin{pmatrix}4a-2\\4d-2\end{pmatrix} = 2\lambda\begin{pmatrix}2a-1\\2d-1\end{pmatrix}$

So $\begin{cases} 2\lambda(2a-1) = 1 \\ 2\lambda(2d-1) = 1 \\ (2a-1)^2 + (2d-1)^2 = 4 \end{cases} \Rightarrow \begin{cases} 4\lambda^2(2a-1)^2 = 1 \\ 4\lambda^2[4-(2a-1)^2] = 1 \end{cases}$

$\Rightarrow 4\lambda^2(2a-1)^2 = 4\lambda^2[4-(2a-1)^2]$

$\Rightarrow 4\lambda^2[2(2a-1)^2 - 4] = 0 \Rightarrow 8\lambda^2(2a-1-\sqrt{2})(2a-1+\sqrt{2}) = 0$

$\Rightarrow \underbrace{\lambda = 0}_{\text{impossible } \because 2\lambda(2a-1)=1\neq 0}$ or $2a-1 = \pm\sqrt{2}$

Since $\lambda \neq 0$, $2\lambda(2a-1) = 1 = 2\lambda(2d-1)$ we have $2a-1 = 2d-1 \Rightarrow a=d$.

- If $a = \frac{1+\sqrt{2}}{2}$ then $\det\begin{pmatrix} a & a-1 \\ a-1 & a \end{pmatrix} = 2a-1 = \sqrt{2}$
- If $a = \frac{1-\sqrt{2}}{2}$ then $\det\begin{pmatrix} a & a-1 \\ a-1 & a \end{pmatrix} = 2a-1 = -\sqrt{2}$

So the matrix with greatest determinant is $\begin{pmatrix} \frac{1+\sqrt{2}}{2} & \frac{-1+\sqrt{2}}{2} \\ \frac{-1+\sqrt{2}}{2} & \frac{1+\sqrt{2}}{2} \end{pmatrix}$