1. [*Colley*, §*4.2 Q3*] Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$$

$$\nabla f = \begin{pmatrix} 2y - 4x \\ 2x - 10y + 4 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$\uparrow$ rearranging & cancelling factor of 2

$$= \frac{1}{10 - 1}\begin{pmatrix} -5 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

So the only critical point of f is at $(2/9, 4/9)$

$$Hf = \begin{pmatrix} -4 & 2 \\ 2 & -10 \end{pmatrix}$$ — it's constant, so this is equal to $Hf(2/9, 4/9)$

$$\mathrm{tr}(Hf) = -14, \quad \det(Hf) = 2^2 \det\begin{pmatrix} -2 & 1 \\ 1 & -5 \end{pmatrix} = 4 \times 9 = 36$$

$$\begin{aligned} \Rightarrow f_{Hf}(\lambda) &= \lambda^2 + 14\lambda + 36 \\ &= (\lambda + 7)^2 - 49 + 36 \\ &= (\lambda + 7)^2 - 13 \end{aligned}$$

So the eigenvalues of Hf are $-7 \pm \sqrt{13}$ which are both negative $\because \sqrt{13} < 7$.

So f has a local maximum at $(2/9, 4/9)$.

2. Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \cos(x)\cos(y)$ lying in the region $-\pi < x < \pi$, $-\pi < y < \pi$.

$$\nabla f = (-\sin x \cos y,\ -\cos x \sin y) = (0,0)$$

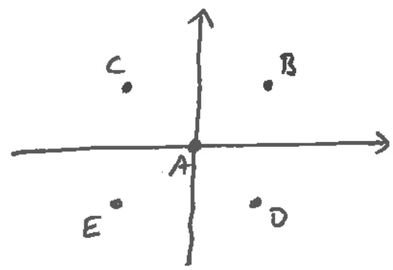
$$\iff \begin{pmatrix} \sin x = 0 \\ \text{or } \cos y = 0 \end{pmatrix} \text{ and } \begin{pmatrix} \cos x = 0 \\ \text{or } \sin y = 0 \end{pmatrix}$$

$$\iff \sin x = \sin y = 0 \quad \text{or} \quad \cos x = \cos y = 0$$

$$\left[\because \text{ can't have } \sin x = \cos x = 0 \text{ or } \sin y = \cos y = 0\right]$$

$$\iff (x,y) = \underbrace{(0,0)}_{A} \text{ or } \underbrace{\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)}_{B} \text{ or } \underbrace{\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right)}_{C} \text{ or } \underbrace{\left(\tfrac{\pi}{2},-\tfrac{\pi}{2}\right)}_{D} \text{ or } \underbrace{\left(-\tfrac{\pi}{2},-\tfrac{\pi}{2}\right)}_{E}$$

$$Hf = \begin{pmatrix} -\cos x \cos y & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y \end{pmatrix}$$



$$= \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{at } A \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{at } B, E \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \text{at } C, D \end{cases}$$

At A : eigenvalues of Hf are $-1, -1$ $\Rightarrow$ <u>local max</u>

At B, C, D, E, $\quad f_{Hf}(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

$\Rightarrow$ eigenvalues of Hf are $1, -1$

$\Rightarrow$ B, C, D, E are <u>saddle points</u>

3. [*Colley, §4.2 Q19*] Find and classify the critical points of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = xy + xz + 2yz + \frac{1}{x}$$

$$\nabla f = \left(y + z - \frac{1}{x^2},\ x + 2z,\ x + 2y \right) = (0,0,0)$$

$$\begin{cases} x + 2z = 0 \\ x + 2y = 0 \end{cases} \Rightarrow y = z = -\frac{x}{2}$$

$$\& \quad y + z - \frac{1}{x^2} = 0 \Rightarrow -x - \frac{1}{x^2} = 0 \Rightarrow x^3 = -1$$
$$\Rightarrow x = -1$$

So the only critical point of f is $(-1, \frac{1}{2}, \frac{1}{2})$.

$$Hf(-1, \tfrac{1}{2}, \tfrac{1}{2}) = \begin{pmatrix} 2/x^3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}\Bigg|_{(-1,\frac{1}{2},\frac{1}{2})} = \underbrace{\begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}}_{A}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix}$$

$$= (-2-\lambda)(\lambda^2 - 4) - (-\lambda - 2) + (2 + \lambda)$$
$$= (\lambda + 2)(-\lambda^2 + 4 - 1 + 1)$$
$$= (\lambda + 2)(-\lambda^2 + 4)$$
$$= (\lambda + 2)(\lambda + 2)(2 - \lambda)$$

So the eigenvalues of Hf at $(-1, \frac{1}{2}, \frac{1}{2})$ are

$-2, -2, 2 \rightarrow$ saddle point.