1. Find the direction(s) in which the function $f(x,y) = e^{x+y}(x^2+y^2)$ is increasing most rapidly at the point $(1,2)$.

$$\nabla f(1,2) = \left(e^{x+y}(x^2+y^2) + 2x e^{x+y} \; , \; e^{x+y}(x^2+y^2) + 2y e^{x+y} \right) \Big|_{(1,2)}$$

$$= e^{x+y}\left(x^2+y^2+2x, \; x^2+y^2+2y \right)\Big|_{(1,2)}$$

$$= e^3 (7, 9)$$

$$\|\nabla f(1,2)\| = e^3\sqrt{49+81} = e^3\sqrt{130}$$

So the direction of most rapid increase is $\frac{1}{\sqrt{130}}(7,9)$

2. Find an equation for the tangent plane to the surface $2x^2+y^2-z^2 = 4$ at the point $(2,0,2)$.

The surface $2x^2+y^2-z^2=4$ is the level surface $(k=4)$ of the function $f(x,y,z) = 2x^2+y^2-z^2$

- $(2,0,2)$ is a pt on the tangent plane
- $\nabla f(2,0,2)$ is $\perp$ to the tangent plane

$$\hookrightarrow = (4x, 2y, -2z)\big|_{(2,0,2)} = (8, 0, -4) = 4(2,0,-1)$$

So the equation of the tangent plane at $(2,0,2)$ is

$$2(x-2) + 0(y-0) + (-1)(z-2) = 0$$

i.e. $\underline{\underline{2x - z = 2}}$

3. Define functions $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathbf{g}(x,y) = \left(\frac{e^{x+y} + e^{x-y}}{2}, \frac{e^{x+y} - e^{x-y}}{2} \right) \quad \text{and} \quad h(s,t) = 2st$$

Find $D_{(-1,2)} f(2,1)$, where $f(x,y) = h(\mathbf{g}(x,y))$.

$$D_{(-1,2)} f(2,1) = \frac{1}{\sqrt{3}} (-1,2) \cdot \nabla f(2,1)$$

$\uparrow \because (-1,2)$ is not a unit vector

By the chain rule,

where $s = \dfrac{e^{x+y} + e^{x-y}}{2}$ & $t = \dfrac{e^{x+y} - e^{x-y}}{2}$

$$Df(1,2) = (f_x, f_y)\Big|_{(1,2)}$$

$$= \left(\frac{\partial h}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial h}{\partial t}\frac{\partial t}{\partial x}, \frac{\partial h}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial h}{\partial t}\frac{\partial t}{\partial y} \right)\Bigg|_{(x,y)=(1,2)}$$

$$= \left(2t \cdot \frac{e^{x+y} + e^{x-y}}{2} + 2s\frac{e^{x+y} - e^{x-y}}{2}, 2s\frac{e^{x+y} - e^{x-y}}{2} + 2t\frac{e^{x+y} + e^{x-y}}{2} \right)\Bigg|_{(2,1)}$$

When $(x,y) = (2,1)$

$s = \dfrac{e^3 + e^1}{2}$

& $t = \dfrac{e^3 - e^1}{2}$

$$= \left((e^3 - e^{+1})(e^3 + e^{+1}) + (e^3 + e^{+1})(e^3 - e^{+1}), (e^3 + e^{+1})(e^3 - e^{+1}) + (e^3 - e^{+1})(e^3 - e^{+1}) \right)$$

$$= 2(e^6 - e^2)(1,1)$$

$$\Rightarrow D_{(-1,2)} f(2,1) = \frac{1}{\sqrt{3}}(-1,2) \cdot 2(e^6 - e^2)(1,1)$$

$$= \frac{2(e^6 - e^2)}{\sqrt{3}}(-1+2)$$

$$= \frac{2(e^6 - e^2)}{\sqrt{3}} \quad \left(= \frac{2e^2}{\sqrt{3}}(e^4 - 1) \right)$$

4. Peeve the guinea pig is standing on a steep Andes mountain. Conveniently, the mountain looks just like the elliptic paraboloid $3x^2+7y^2+5z=20$. Owing to her short legs, the steepest grade that Peeve can climb up the mountain is $\frac{1}{5}$.

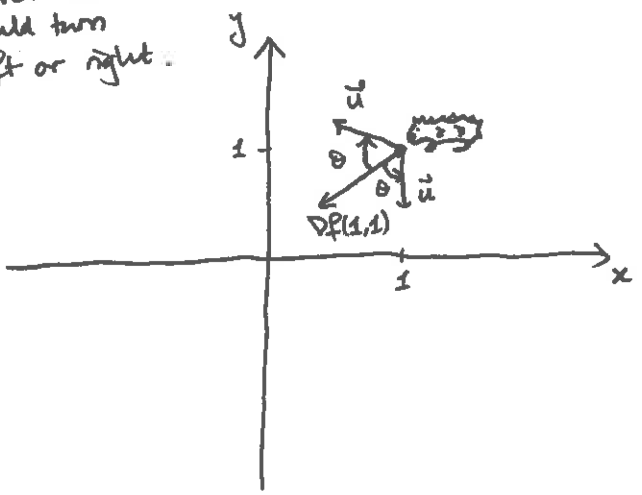
Given that Peeve's (x,y)-coordinates are $(1,1)$ and she is facing in the direction of steepest ascent, find the smallest angle that Peeve must turn in order to be able to ascend the mountain.

$$3x^2+7y^2+5z=20 \iff z = 4 - \frac{3}{5}x^2 - \frac{7}{5}y^2$$

So the mountain is the graph of $f(x,y) = 4 - \frac{3}{5}x^2 - \frac{7}{5}y^2$.

Let $\vec{u}$ be a direction* that Peeve can ascend with grade $\frac{1}{5}$ — note that Peeve is currently facing in the direction of $\nabla f(1,1)$. Let θ be the angle between $\vec{u}$ and $\nabla f(1,1)$. Then

*Note: Peeve could turn left or right.



$$\frac{1}{5} = D_{\vec{u}} f(1,1) = \|\nabla f(1,1)\| \cos\theta$$

$$\nabla f(1,1) = -\frac{2}{5}(3x, 7y)\Big|_{(1,1)}$$

$$= -\frac{2}{5}(3,7)$$

$$\Rightarrow \|\nabla f(1,1)\| = \frac{2}{5}\sqrt{9+49} = \frac{2\sqrt{58}}{5}$$

$$\Rightarrow \cos\theta = \frac{1}{5\|\nabla f(1,1)\|} = \frac{1}{2\sqrt{58}}$$

$$\Rightarrow \theta = \arccos\frac{1}{2\sqrt{58}} \approx 1.51 \text{ radians} \approx 86.2^\circ$$

So Peeve must turn ≈ 1.51 rads / 86.2° in either direction to be able to ascend the mountain.