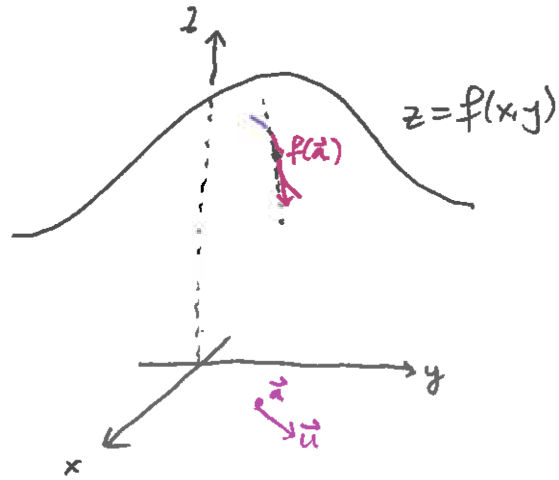
Math 290-2 Class 20

Monday 25th February 2019

Directional derivatives



The **directional derivative** $D_{\mathbf{u}}f(\mathbf{a})$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{a}$ in a given direction (unit vector) $\mathbf{u}$ is defined by

If $\vec{v}$ is not a unit vector, then $D_{\vec{v}}f$ means $D_{(\vec{v}/\|\vec{v}\|)}f$.

$$D_{\mathbf{u}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a}+h\mathbf{u}) - f(\mathbf{a})}{h}$$

Notice that for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$$f_x(a,b) = D_{\mathbf{i}}f(a,b) \text{ and } f_y(a,b) = D_{\mathbf{j}}f(a,b)$$

Fun fact: if f is differentiable at $\mathbf{a}$, then $\boxed{D_{\mathbf{u}}f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a})}$ (so '$D_{\mathbf{u}} = \mathbf{u} \cdot \nabla$'). In particular:

$$\boxed{D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta}$$

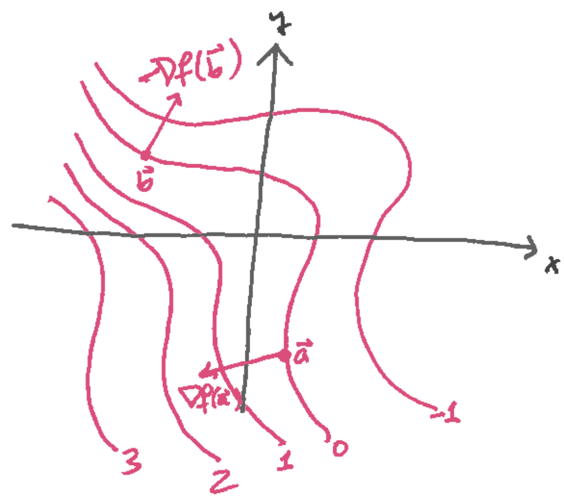
where θ is the angle between $\nabla f(\mathbf{a})$ and $\mathbf{u}$ (with $0 \leqslant \theta \leqslant \pi$).

Some fun consequences:

(i) $D_{\mathbf{u}}f(\mathbf{a})$ is maximised when $\mathbf{u}$ points in the same direction as $\nabla f(\mathbf{a})$—thus $\nabla f(\mathbf{a})$ points in the direction of fastest increase of f;

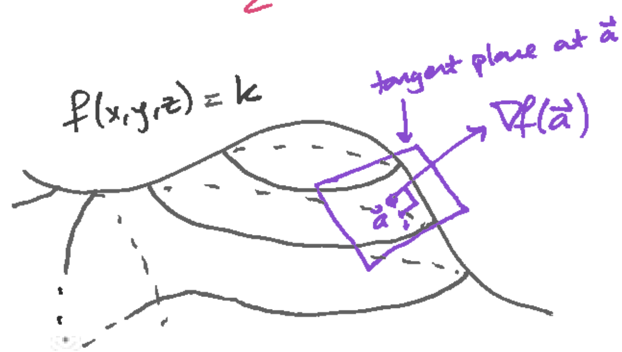
(ii) $D_{\mathbf{u}}f(\mathbf{a})$ is minimised when $\mathbf{u}$ points in the opposite direction from $\nabla f(\mathbf{a})$—thus $-\nabla f(\mathbf{a})$ points in the direction of fastest decrease of f;

(iii) $D_{\mathbf{u}}f(\mathbf{a}) = 0$ when $\mathbf{u} \perp \nabla f(\mathbf{a})$.



In fact, (iii) implies that:

- For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point (a,b), the vector $\nabla f(a,b)$ is perpendicular to (the tangent line to) the level curve of f at (a,b);
- For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a point (a,b,c), the vector $\nabla f(a,b,c)$ is perpendicular to (the tangent plane to) the level surface of f at (a,b,c).



1. Compute the directional derivative of the function $f(x,y) = x^2y + y^2x$ at $(1,2)$ in the direction of the vector $(-2,1)$.

$$\nabla f(1,2) = (2xy + y^2,\ x^2 + 2xy)\Big|_{(1,2)} = (8,5)$$

Unit vector $\parallel$ to $(-2,1)$: $\frac{1}{\sqrt{5}}(-2,1)$

$$\Rightarrow D_{(-2,1)} f(1,2) = \frac{1}{\sqrt{5}}(-2,1)\cdot(8,5) = \frac{1}{\sqrt{5}}(-16+5)$$

$$= -\frac{11}{\sqrt{5}}$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function, let $\mathbf{u} = (u,v)$ be a unit vector in $\mathbb{R}^2$, and let $\mathbf{a} = (a,b)$ in $\mathbb{R}^2$. Assuming f is differentiable at $\mathbf{a}$, use the chain rule to show that $D_{\mathbf{u}} f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a})$.

$$D_{\vec{u}} f(\vec{a}) = \lim_{t\to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} = \frac{d}{dt} f(\underbrace{\vec{a} + t\vec{u}}_{\begin{cases} x = a + tu \\ y = b + tv \end{cases}})\Big|_{t=0}$$

$$= \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right]_{t=0}$$

$$= \frac{\partial f}{\partial x}(a,b)\cdot u + \frac{\partial f}{\partial y}(a,b)\cdot v$$

$$= \nabla f(a,b)\cdot(u,v)$$

$$= \nabla f(\vec{a})\cdot\vec{u}.$$

3. Find the direction(s) in which the function $f(x,y) = e^{x+y}(x^2+y^2)$ is increasing most rapidly at the point $(1,2)$.

$$\nabla f(1,2) = \left(e^{x+y}(x^2+y^2) + 2xe^{x+y},\ e^{x+y}(x^2+y^2) + 2ye^{x+y}\right)\Big|_{(1,2)}$$

$$= e^{x+y}\left(x^2+y^2+2x,\ x^2+y^2+2y\right)\Big|_{(1,2)}$$

$$= e^3(1+4+2,\ 1+4+4)$$

$$= e^3(7,9)$$

$$\|\nabla f(1,2)\| = e^3\sqrt{49+81} = e^3\sqrt{130}$$

$\Rightarrow$ direction of greatest increase $= \underline{\underline{\frac{1}{\sqrt{130}}(7,9)}}$

4. Find an equation for the tangent plane to the surface $2x^2+y^2-z^2 = 4$ at the point $(2,0,2)$.

This is the level surface to the function $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = 2x^2+y^2-z^2$ at $(2,0,2)$

$\Rightarrow \underbrace{\nabla f(2,0,2)}$ is $\perp$ to tangent plane at $(2,0,2)$

$$= (4x, 2y, -2z)\Big|_{(2,0,2)} = (8,0,-4)$$

$$= 4(2,0,-1)$$

So $\begin{cases} (2,0,2) \text{ is a pt on the plane} \\ (2,0,-1) \text{ is } \perp \text{ to the plane} \end{cases}$

$\Rightarrow$ Eq of plane $= 2(x-2) + 0(y-0) + (-1)(z-2) = 0$

Equivalently: $\underline{\underline{2x - z = 2}}$