1. Compute the gradient vector of the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(x,y) = \sin(x^2+y^2)e^{xy}$$

Let $f(x,y) = \sin(x^2+y^2)$ & $g(x,y) = e^{xy}$

$$\begin{aligned}
\nabla f(x,y) &= g(x,y)\,\nabla f(x,y) + f(x,y)\,\nabla g(x,y) \\
&= e^{xy}\left(2x\cos(x^2+y^2),\ 2y\cancel{\cos}(x^2+y^2)\right) \\
&\quad + \sin(x^2+y^2)\left(ye^{xy},\ xe^{xy}\right) \\
&= e^{xy}\left(2x\cos(x^2+y^2) + y\sin(x^2+y^2),\ 2y\cos(x^2+y^2) + x\sin(x^2+y^2)\right)
\end{aligned}$$

2. Compute $\nabla(\mathbf{f}\cdot\mathbf{g})$, where $\mathbf{f},\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$\mathbf{f}(x,y) = (x^2+y^2, x^2-y^2) \quad \text{and} \quad \mathbf{g}(x,y) = (2xy, -2xy)$$

$$\begin{aligned}
\frac{\partial(\vec{f}\cdot\vec{g})}{\partial x}(x,y) &= \vec{f}_x\cdot\vec{g} + \vec{f}\cdot\vec{g}_x = (2x,2x)\cdot(2xy,-2xy) + \\
&\qquad (x^2+y^2,\ x^2-y^2)\cdot(2y,-2y) \\
&= 0 + 2x^2y + 2y^3 - 2x^2y + 2y^3 = 4y^3
\end{aligned}$$

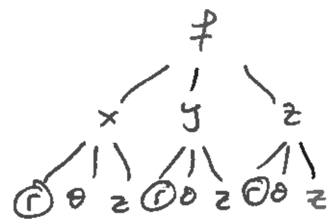
$$\begin{aligned}
\frac{\partial(\vec{f}\cdot\vec{g})}{\partial y}(x,y) &= \vec{f}_y\cdot\vec{g} + \vec{f}\cdot\vec{g}_y = (2y,-2y)\cdot(2xy,-2xy) + \\
&\qquad (x^2+y^2,\ x^2-y^2)\cdot(2x,-2x) \\
&= (4xy^2 + 4xy^2) + (2x^3 + 2xy^2 - 2x^3 + 2xy^2) \\
&= 12xy^2
\end{aligned}$$

$$\Rightarrow \nabla(\vec{f}\cdot\vec{g})(x,y) = (4y^3,\ 12xy^2) \qquad (= 4y^2(y,3x))$$

3. Given that $f(x,y,z) = x^2e^y + y^2e^z + z^2e^x$, compute the rate of change of $f(\mathbf{x})$ with respect to the distance of $\mathbf{x}$ from the z-axis when $\vec{x} = (1,1,2)$.

The distance of $\vec{x}$ from the z-axis is r, where

$$\vec{x}(r,\theta,z) = (r\cos\theta,\ r\sin\theta,\ z)$$



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$= (2xe^y + z^2e^x)(\cos\theta) + (2ye^z + x^2e^x)(\sin\theta) + \frac{\partial f}{\partial z}\cdot 0$$

At $(1,1,2)$, $\theta = \frac{\pi}{4} \Rightarrow \cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$

So $$\frac{\partial f}{\partial r}(1,1,2) = \frac{1}{\sqrt{2}}\left(2\cdot 1\cdot e^1 + 2^2\cdot e^1 + 2\cdot 1\cdot e^2 + 1^2e^1\right)$$

$$= \frac{e}{\sqrt{2}}(7+2e)$$

4. Find the Jacobian matrix of $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the variables (s,t), where

$$\mathbf{f}(x,y) = (2xy, x^2-y^2), \quad x(s,t) = e^s\cos t \quad \text{and} \quad y(s,t) = e^s\sin t$$

$$D\vec{f}(s,t) = (D\vec{f}(x,y))\,(D\vec{x}(s,t))$$

$$= \begin{pmatrix} 2y & 2x \\ 2x & -2y \end{pmatrix}\begin{pmatrix} e^s\cos t & -e^s\sin t \\ e^s\sin t & e^s\cos t \end{pmatrix}$$

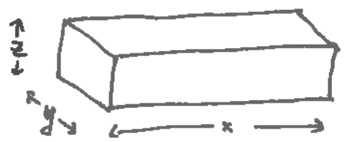
$$= \begin{pmatrix} 2e^s\sin t & 2e^s\cos t \\ 2e^s\cos t & -2e^s\sin t \end{pmatrix}\begin{pmatrix} e^s\cos t & -e^s\sin t \\ e^s\sin t & e^s\cos t \end{pmatrix}$$

$$= 2e^{2s}\begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$= 2e^{2s}\begin{pmatrix} 2\sin t\cos t & \cos^2 t - \sin^2 t \\ \cos^2 t - \sin^2 t & -2\sin t\cos t \end{pmatrix}$$

$$= 2e^{2s}\begin{pmatrix} \sin 2t & \cos 2t \\ \cos 2t & -\sin 2t \end{pmatrix}$$

5. [Colley, §2.5 Q6] A rectangular stick of bvutter is placed in a microwave oven to melt. When the butter's length is 6 in and its square cross section on one side measures $\frac{3}{2}$ in, its length is decreasing at a rate of $\frac{1}{4}$ in min^{-1}, and its cross-sectional edge is decreasing at a rate of $\frac{1}{8}$ in min^{-1}. How fast is butter melting at that instant (i.e. what is the rate of decrease of its volume in in^3 min^{-1})?



Volume $= V = xyz$

We need to find $\frac{dV}{dt}$ when $x=6$ and $y = z = \frac{3}{2}$.

We're told $\frac{dx}{dt} = -\frac{1}{4}$ and $\frac{dy}{dt} = \frac{dz}{dt} = -\frac{1}{8}$ at $(6, \frac{3}{2}, \frac{3}{2})$

$$\Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

$$= yz \cdot \left(-\frac{1}{4}\right) + xz \cdot \left(-\frac{1}{8}\right) + xy \cdot \left(-\frac{1}{8}\right)$$

$$= -\frac{9}{4} \cdot \frac{1}{4} - 9 \cdot \frac{1}{8} - 9 \cdot \frac{1}{8}$$

$$= -\frac{1}{16}\left(9 + 18 + 18\right) = -\frac{45}{16}$$

So the volume decreases at $\frac{45}{16}$ in^3 min^{-1}.

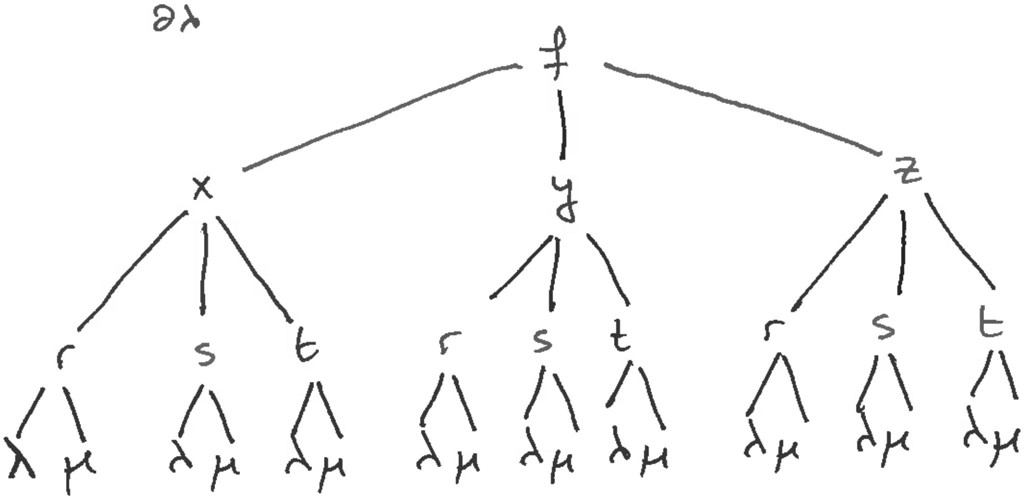
6. Suppose $f = f(x,y,z)$, $(x,y,z) = \mathbf{x}(r,s,t)$ and $(r,s,t) = \mathbf{r}(\lambda,\mu)$.
Find expressions for f_λ and f_μ.

$$Df(\lambda,\mu) = (Df(\vec{x}))(D\vec{x}(\vec{r}))(D\vec{r}(\lambda,\mu))$$

$$= \begin{pmatrix} f_x & f_y & f_z \end{pmatrix} \begin{pmatrix} x_r & x_s & x_t \\ y_r & y_s & y_t \\ z_r & z_s & z_t \end{pmatrix} \begin{pmatrix} r_\lambda & r_\mu \\ s_\lambda & s_\mu \\ t_\lambda & t_\mu \end{pmatrix}$$

$$= \begin{pmatrix} f_x & f_y & f_z \end{pmatrix} \begin{pmatrix} x_r r_\lambda + x_s s_\lambda + x_t t_\lambda & x_r r_\mu + x_s s_\mu + x_t t_\mu \\ y_r r_\lambda + y_s s_\lambda + y_t t_\lambda & y_r r_\mu + y_s s_\mu + y_t t_\mu \\ z_r r_\lambda + z_s s_\lambda + z_t t_\lambda & z_r r_\mu + z_s s_\mu + z_t t_\mu \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{\begin{matrix} f_x x_r r_\lambda + f_x x_s s_\lambda + f_x x_t t_\lambda \\ + f_y y_r r_\lambda + f_y y_s s_\lambda + f_y y_t t_\lambda \\ + f_z z_r r_\lambda + f_z z_s s_\lambda + f_z z_t t_\lambda \end{matrix}}_{= \frac{\partial f}{\partial \lambda}} & \underbrace{\begin{matrix} f_x x_r r_\mu + f_x x_s s_\mu + f_x x_t t_\mu \\ + f_y y_r r_\mu + f_y y_s s_\mu + f_y y_t t_\mu \\ + f_z z_r r_\mu + f_z z_s s_\mu + f_z z_t t_\mu \end{matrix}}_{= \frac{\partial f}{\partial \mu}} \end{pmatrix}$$



One term for each path from f to λ (resp. f to μ) in the tree. E.g. the path $f \to y \to t \to \lambda$ corresponds to the term $f_y y_t t_\lambda = \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \frac{\partial t}{\partial \lambda}$