Math 290-2 Class 15

Wednesday 13th February 2019

Limits

Consider a function f from (some subset of) $\mathbb{R}^n$ to $\mathbb{R}$. Given a vector $\mathbf{a}$ in $\mathbb{R}^n$, the **limit** of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{a}$, if it exists, is the value ℓ that the function becomes arbitrary close to whenever $\mathbf{x}$ is an arbitrarily small positive distance from $\mathbf{a}$. In this case, we write

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \ell \quad \text{or} \quad f(\mathbf{x}) \to \ell \text{ as } \mathbf{x} \to \mathbf{a}$$

Limits do not always exist.[a] Some ways that limits can fail to exist include:

- The usual '1-dimensional' reasons, such as the denominator of a fraction tending to zero while its numerator does not.
- The function might approach multiple values depending on the 'path' along which the variable $\mathbf{x}$ approaches $\mathbf{a}$. If this is the case, a limit does not exist.

For example, if you suspect a limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$ because the limit is not 'independent of path', some suggestions include:

- Set $x = 0$ and compute the limit as $y \to 0$, and set $y = 0$ and compute the limit as $x \to 0$.
- Set $y = mx$ for some real number m and compute the limit as $x \to 0$.
- Set $y = x^k$ for some power k and compute the limit as $x \to 0$.

If any of the above limits do not equal any of the others, the limit does not exist.

As a rule of thumb, if f is built out of nice, continuous functions (such as polynomials, exponentials and trig functions) using arithmetic operations, and the denominators involved do not tend to zero, then a limit exists. If not, some more care is needed.

If you're struggling to compute a limit (or show it doesn't exist), try converting to a different system of coordinates, such as polar coordinates (in $\mathbb{R}^2$), or cylindrical or spherical coordinates (in $\mathbb{R}^3$).

A related concept is *continuity*:

- The limit $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ might not actually be equal to $f(\mathbf{a})$.
- If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$, we say f is **continuous at a**.
- If f is continuous at $\mathbf{a}$ for all $\mathbf{a}$ in its domain, we say f is **continuous**.

[a] `https://youtu.be/oDAKKQuBtDo?t=45`

1. For each of the following, either evaluate the limit or show it does not exist.

(a) $\lim_{(x,y,z)\to(0,1,\pi)} \dfrac{e^{x+y^2}\cos(x^2+2z)+xyz}{x^2+y^2+z^2}$.

Note: all fns are continuous & well defined near $(0,1,\pi)$ and the denominator does not tend to zero.

$\Rightarrow$ the limit is $\dfrac{e^{0+1^2}\cos(0^2+2\pi)+0\cdot 1\cdot\pi}{0^2+1^2+\pi^2}$

$$= \frac{e}{1+\pi^2}$$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{2x^2+3y^2}{x^2+y^2}$

Consider the paths $P_1 : (x,y) = (t,0)$ and $P_2 : (x,y) = (0,t)$

- limit along P_1 as $(x,y)\to(0,0)$: $\lim_{t\to 0} \dfrac{2t^2+3\cdot 0^2}{t^2+0^2} = \lim_{t\to 0} 2 = 2$
- limit along P_1 as $(x,y)\to(0,0)$: $\lim_{t\to 0} \dfrac{2\cdot 0^2+3\cdot t^2}{0^2+t^2} = \lim_{t\to 0} 3 = 3$

$\Rightarrow$ Limit does not exist.

(c) $\lim_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$ [*Colley*, §2.2 *Q15*]

$$x^4-y^4 = (x^2-y^2)(x^2+y^2)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0.$$

(d) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Limit along x-axis $[(x,y) = (t,0)]$: $\lim_{t\to 0} \frac{0}{t^2+0^2} = 0$

Limit along y-axis $[(x,y) = (0,t)]$: $\lim_{t\to 0} \frac{0}{0^2+t^2} = 0$

Hmm .....

Limit along $y=x$ $[(x,y) = (t,t)]$: $\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$

So the limit does not exist.

(e) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Limit along $y=mx$ for any m $[(x,y) = (t,mt)]$: $\lim_{t\to 0} \frac{m^2t^3}{t^2+m^4t^4} = \lim_{t\to 0} \frac{m^2t}{1+m^4t^2} = 0$

Hmm ....

Intuition: make the powers of x match the powers of y.

Limit along $x = y^2$ $[(x,y) = (t^2,t)]$: $\lim_{t\to 0} \frac{t^4}{t^4+t^4} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$

So the limit does not exist.

(f) $\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$ [*Colley*, §2.2 *Q30*]

One approach: take limit along $y = mx$:

$$\lim_{t\to 0} \frac{t^2 + mt^2 + m^2t^2}{t^2 + m^2t^2} = \lim_{t\to 0} \frac{1+m+m^2}{1+m^2} = \begin{cases} 1 & \text{if } m=0 \\ \frac{3}{2} & \text{if } m=1 \end{cases}$$

Another approach: Convert to polar coordinates:

$$\lim_{r\to 0} \frac{r^2 + r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} \left(1 + \underline{\cos\theta\sin\theta}\right)$$

↳ Depends on θ

eg/ limit $= 1$ if $\theta = 0$, or $\frac{3}{2}$ if $\theta = \frac{\pi}{4}$

So the limit does not exist

(g) $\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{\sqrt{x^2+y^2}}$

In polar coordinates:

$$\lim_{r\to 0} \frac{r^2 + r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} \left(r\left(1+\cos\theta\sin\theta\right)\right)$$

$$= \underline{\underline{0}} \quad \left(\text{regardless of how we vary } \theta\right)$$