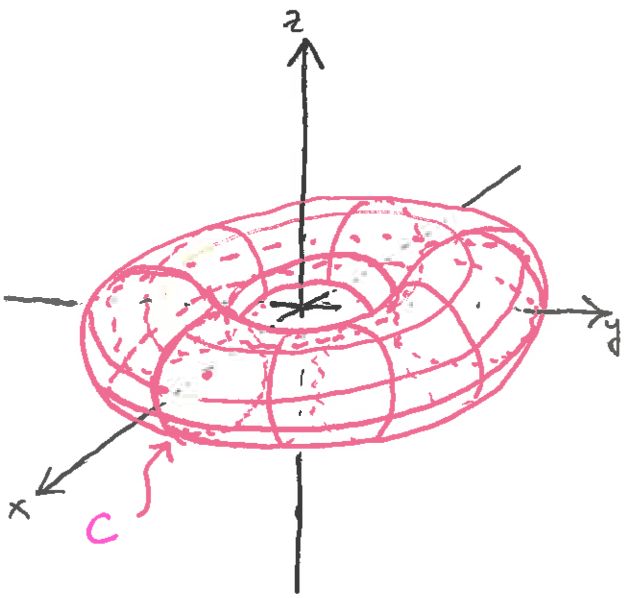
1. Let C be the circle in the (x,z) plane with centre $(2,0,0)$ and radius 1. Find the cylindrical coordinate equation of the torus traced by C upon rotation by 2π radians about the z-axis.



Rotational symmetry about z-axis
$\Rightarrow$ independent of θ.

In (x,z)-plane, $r = x$, and the circle is described by $(x-2)^2 + z^2 = 1$

So the equation of the surface in cylindrical polar coordinates is:

$$\underline{\underline{(r-2)^2 + z^2 = 1}}$$

2. Find the cartesian equation of the surface described in spherical coordinates by

$$\rho^2(a\sin^2\varphi\cos^2\theta + b\sin^2\varphi\sin^2\theta + c\cos^2\theta) = 1$$

Sketch this surface in when $a > b > c > 0$.

Note $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\theta$

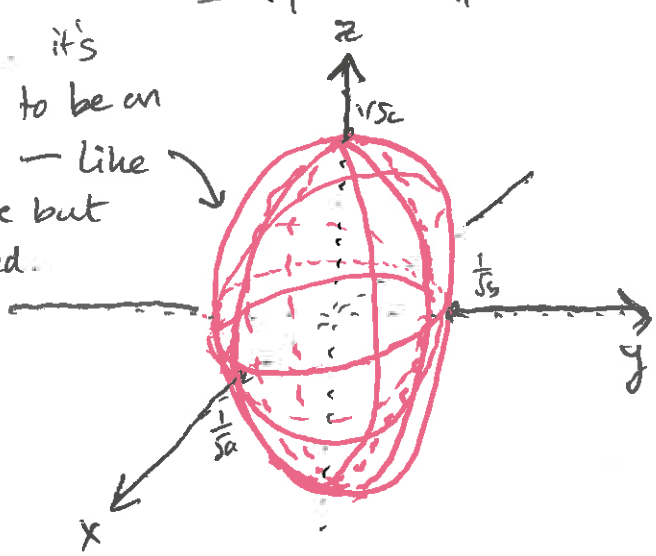
So in cartesian coordinates, the surface is described by

$$ax^2 + by^2 + cz^2 = 1.$$

$$\equiv \left(\frac{x}{p}\right)^2 + \left(\frac{y}{q}\right)^2 + \left(\frac{z}{r}\right)^2 = 1 \quad \text{where } p = \frac{1}{\sqrt{a}},\ q = \frac{1}{\sqrt{b}},\ r = \frac{1}{\sqrt{c}}$$

when $a > b > c > 0$
$(\Rightarrow p < q < r)$

Eww... it's supposed to be an ellipsoid — like a sphere but squished.



x-intercepts: $\pm\frac{1}{\sqrt{a}}$

y-intercepts: $\pm\frac{1}{\sqrt{b}}$

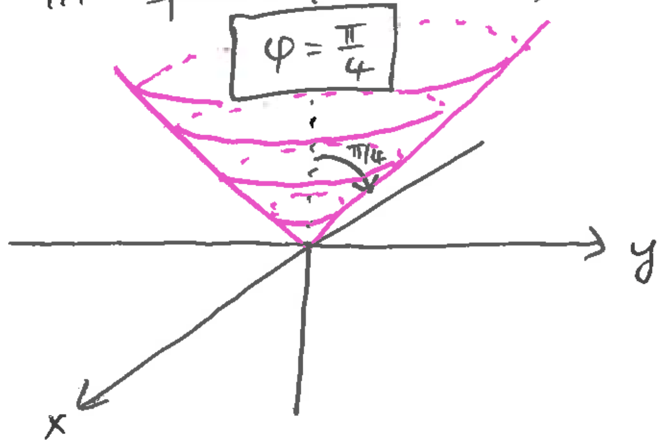
z-intercepts: $\pm\frac{1}{\sqrt{c}}$

3. Let S be the cone whose cylindrical equation is $z = r$, $z \geqslant 0$. Describe S in spherical coordinates.

$$z = r \Rightarrow z^2 = r^2 = x^2 + y^2$$
$$\Rightarrow \rho^2 \cos^2\varphi = \rho^2 \sin^2\varphi \, (\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\varphi$$
$$\Rightarrow \cos\varphi = \pm \sin\varphi \Rightarrow \varphi = \frac{\pi}{4} \text{ or } \underbrace{\frac{3\pi}{4}}_{\text{impossible } \because z \geq 0}$$

So in spherical coordinates, S is described by

$$\boxed{\varphi = \frac{\pi}{4}}$$



4. Find a way of converting between cylindrical and spherical coordinates.

Note: $\theta = \theta$.

Like above, $r^2 = x^2 + y^2 = \rho^2 \sin^2\varphi$ & $z = \rho\cos\varphi$

$$\Rightarrow r = \rho \sin\varphi$$

(Note: we allow $r < 0$ & $\rho < 0$)

So we get the following conversions:

<u>Sph $\rightarrow$ Cyl</u>

$$\begin{cases} r = \rho \sin\varphi \\ \theta = \theta \\ z = \rho\cos\varphi \end{cases}$$

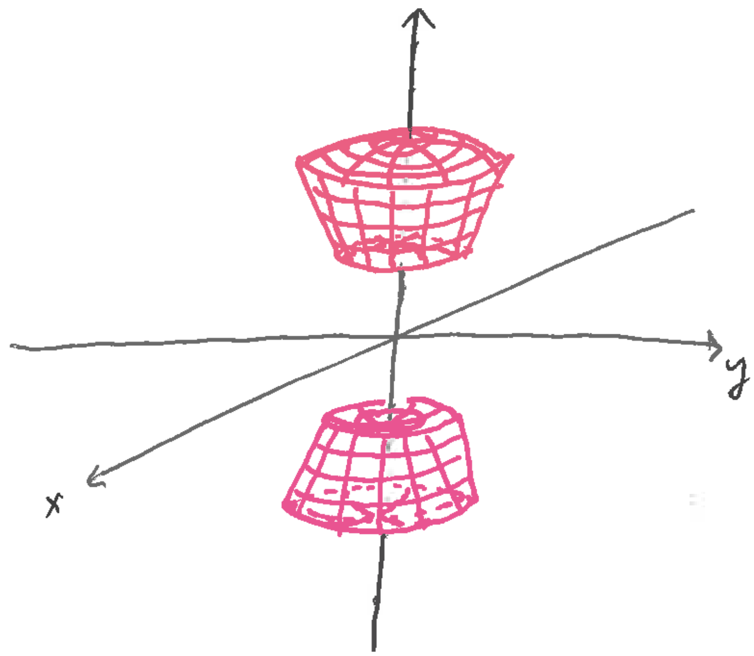
<u>Cyl $\rightarrow$ Sph</u>

$$\begin{cases} \rho^2 = r^2 + z^2 \\ \tan\varphi = \dfrac{r}{z} \\ \theta = \theta \end{cases}$$

5. Sketch the solid region of $\mathbb{R}^3$ described in spherical coordinates by

$$1 \leqslant \rho^2 \leqslant 4, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{6}, \quad 0 \leqslant \theta \leqslant 2\pi$$

and describe this region in cartesian and cylindrical coordinates.



<u>Cylindrical</u>

$0 \le \varphi \le \frac{\pi}{6} \Rightarrow 0 \le \tan\varphi \le \frac{1}{\sqrt{3}} \Rightarrow 0 \le r \le \frac{1}{\sqrt{3}} z$

$1 \le \rho^2 \le 4 \Rightarrow 1 \le r^2 + z^2 \le 4$

So it is described in cylindrical coordinates by

$$\boxed{0 \le r \le \frac{1}{\sqrt{3}} z, \quad 1 \le r^2 + z^2 \le 4, \quad 0 \le \theta \le 2\pi}$$

<u>Cartesian</u>

$0 \le \tan\varphi \le \frac{1}{\sqrt{3}} \Rightarrow 0 \le \sqrt{x^2+y^2} \le \frac{1}{\sqrt{3}} z \Rightarrow 0 \le x^2 + y^2 \le \frac{z^2}{3}$

$1 \le \rho^2 \le 4 \Rightarrow 1 \le x^2 + y^2 + z^2 \le 4$

$0 \le \theta \le 2\pi \Rightarrow$ no constraint

So it is described in cartesian coordinates by

$$\boxed{0 \le x^2 + y^2 \le \frac{z^2}{3}, \quad 1 \le x^2 + y^2 + z^2 \le 4}$$