1. Let **a** and **b** be two vectors in $\mathbb{R}^3$. Expand the following expression and simplify it as much as possible.

$$(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$$

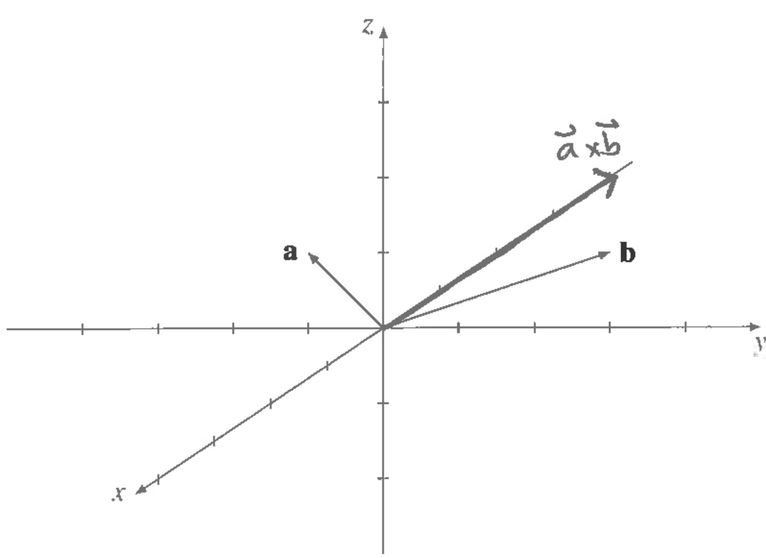
$$\begin{aligned}
&(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) && \text{distributivity} \\
&= \vec{a} \times (\vec{a}-\vec{b}) + \vec{b} \times (\vec{a}-\vec{b}) && \text{distributivity \& linearity} \\
&= \vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b} && \vec{a} \parallel \vec{a} \text{ and } \vec{b} \parallel \vec{b} \\
&= -\vec{a} \times \vec{b} + \vec{b} \times \vec{a} && \text{anticommutativity} \\
&= -\vec{a} \times \vec{b} - \vec{a} \times \vec{b} && \text{simplifying} \\
&= -2(\vec{a} \times \vec{b})
\end{aligned}$$

2. Sketch $\mathbf{a} \times \mathbf{b}$ in the following diagram, where **a** and **b** are in the (y, z)-plane.

$$\|\vec{a}\| = \sqrt{1+1} = \sqrt{2}$$

$$\|\vec{b}\| = \sqrt{9+1} = \sqrt{10}$$

$$\begin{aligned}
\sin\theta &= \sqrt{1-\cos^2\theta} \\
&= \sqrt{1-\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)^2} \\
&= \sqrt{1-\frac{4}{20}} \\
&= \sqrt{1-\frac{1}{5}} \\
&= \frac{2}{\sqrt{5}}
\end{aligned}$$



$$\vec{a} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{a}\cdot\vec{b} = -3+1 = -2$$

$$\Rightarrow \|\vec{a}\times\vec{b}\| = \sqrt{2}\cdot\sqrt{10}\cdot\frac{2}{\sqrt{5}} = 2\cdot\sqrt{\frac{20}{5}} = 4$$

right-hand rule

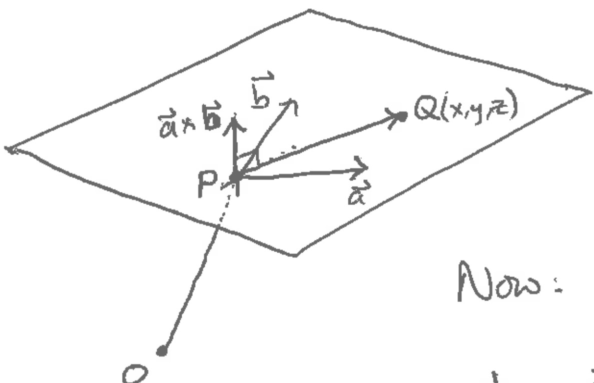
$$\Rightarrow \vec{a}\times\vec{b} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$$

3. Evaluate $(2,1,-1)\times(1,0,3)$.

$$\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix} = \underbrace{\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}}_{=3}\vec{i} - \underbrace{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}_{=8}\vec{j} + \underbrace{\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}}_{=-1}\vec{k}$$

$$= (3, 8, -1)$$

4. Find the equation of the plane that passes through the point $P(-1,3,2)$ and is parallel to the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$.



$Q(x,y,z)$ is in the plane

$\iff \overrightarrow{PQ}$ is in span$\{\vec{a},\vec{b}\}$

$\iff \overrightarrow{PQ}$ is perpendicular to $\vec{a}\times\vec{b}$

Now: $\vec{a}\times\vec{b} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix}$

& $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} x+1 \\ y-3 \\ z-2 \end{pmatrix}$

So $Q(x,y,z)$ is in the plane

$\iff \begin{pmatrix} x+1 \\ y-3 \\ z-2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix} = 0$

$\iff -2(x+1) - (y-3) - 4(z-2) = 0$

$\iff \underline{\underline{2x + y + 4z = 9}}$

5. For each of the following statements about vectors **a**, **b**, and **c** in $\mathbb{R}^3$, determine whether it is always, sometimes or never true.

(a) $\|\mathbf{a} \times \mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2$

Always Let $0 \le \theta < \pi$ be the angle between $\vec{a}$ and $\vec{b}$

$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 \sin^2\theta$, $\quad (\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2\|\vec{b}\|^2 \cos^2\theta$

$\Rightarrow \quad \|\vec{a} \times \vec{b}\|^2 + (\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2\|\vec{b}\|^2(\cos^2\theta + \sin^2\theta) = \|\vec{a}\|^2\|\vec{b}\|^2$

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

Sometimes

- True when $\vec{a} = \vec{b} = \vec{c} = \vec{0}$ (both expressions $= \vec{0}$)
- False when $\vec{a} = \vec{b} = \vec{i}$ and $c = \vec{j}$

$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$

$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0} \neq -\vec{j}$

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

Always [a.k.a. Jacobi's identity]

The proof is tedious without more convenient notation conventions.

(d) Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is an orthonormal basis of $\mathbb{R}^3$. Then $\mathbf{a} \times \mathbf{b} = \mathbf{c}$.

Sometimes

- True when $\vec{a}, \vec{b}, \vec{c} = \vec{i}, \vec{j}, \vec{k}$ — then $\vec{i} \times \vec{j} = \vec{k}$
- False when $\vec{a}, \vec{b}, \vec{c} = \vec{i}, \vec{j}, -\vec{k}$ — then $\vec{i} \times \vec{j} = \vec{k} \neq -\vec{k}$.