1. For each of the following statements, determine whether it is always, sometimes or never true.

(a) The matrix of orthogonal projection onto a subspace of $\mathbb{R}^n$ is symmetric.

<u>Always</u>

If Q is the mx whose columns are an orthonormal basis of the subspace, then orthogonal projection onto the subspace has matrix QQ^T. And $(QQ^T)^T = (Q^T)^T Q^T = QQ^T$.

Another argument: let $\vec{v}_1 \dots \vec{v}_k$ be an orthonormal basis of the subspace and let $\vec{v}_{k+1} \dots \vec{v}_n$ be an orthonormal basis of its orthogonal complement. Then $\vec{v}_1 \dots \vec{v}_n$ is an o.n. basis of eigenvectors!

(b) Let a, b and c be real numbers. Then $\begin{pmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{pmatrix}$ is orthogonally diagonalisable.

<u>Sometimes</u>

- <u>True</u> if $a=b=c=0 \Rightarrow$ the matrix is I_3 which is evidently orthogonally diagonalisable.
- <u>False</u> if $a=b=c=1 \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$\Rightarrow$ the mx isn't symmetric $\Rightarrow$ it's not orthogonally diagonalisable.

(c) Let A be a symmetric matrix. Then A is invertible.

<u>Sometimes</u>

- <u>True</u> when $A = I_2 \Rightarrow A^{-1} = I_2 = A^T = A$
- <u>False</u> when $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow A^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A$ but A is not invertible.

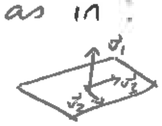
(d) Let A be a matrix such that if $\vec{v}$ and $\vec{w}$ are eigenvectors of A with distinct eigenvalues, then $\vec{v}$ is perpendicular to $\vec{w}$. Then A is orthogonally diagonalisable.

<u>Sometimes</u> <u>True</u> when $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then the eigenspaces of A are $E_1 = \text{span}\{\vec{e_1}\}$, $E_2 = \text{span}\{\vec{e_2}\}$, $E_3 = \text{span}\{\vec{e_3}\}$
$\Rightarrow \vec{e_1}, \vec{e_2}, \vec{e_3}$ is an o.n. basis of eigenvectors.

<u>False</u> when $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, then A is not diagonalisable since $GM(1) = 2$ but $A - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow GM(1) = 1 < 2$.
But $E_1 = \text{span}\{\vec{e_1}\}$ & $E_2 = \text{span}\{\vec{e_3}\}$ so any distinct e.vals have perpendicular eigenvectors.

(e) Let A be the matrix of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by rotation by θ radians about the line spanned by $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, where $0 < \theta < \pi$. Then A is symmetric.

<u>Never</u> Let $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ & let $\vec{v_2}, \vec{v_3}$ be orthogonal vectors in the plane perp. to $\vec{v_1}$, as in



Then A is similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
(this is the mx of A w.r.t. the basis $\vec{v_1}, \vec{v_2}, \vec{v_3}$).

$$\Rightarrow f_A(\lambda) = (\lambda - 1)(\lambda^2 - 2\cos\theta + 1)$$

But $(2\cos\theta)^2 - 4 \times 1 \times 1 = 4(\cos^2\theta - 1) < 0$
since $|\cos\theta| < 1$ for $0 < \theta < \pi$

$\Rightarrow f_A(\lambda)$ can't be completely factorised
$\Rightarrow A$ is not diagonalisable
$\Rightarrow A$ is not symmetric.

2. Find an orthogonal matrix S such that $S^T AS$ is diagonal, where $A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$.

$f_A(\lambda) = \lambda^2 - 0 \cdot \lambda + (-16-9) = \lambda^2 - 25 = (\lambda-5)(\lambda+5)$

- $A - 5I = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \Rightarrow E_5 = \text{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$
- $A + 5I = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \Rightarrow E_{-5} = \text{span}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$

$\Rightarrow \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an orthonormal basis of eigenvectors of A.

$\Rightarrow S$ is orthogonal, where $S = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$

and $S^TAS = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$ is diagonal.

Rephrased in actual handout

3. Let A be a non-identity, symmetric, orthogonal 2×2 matrix. Show that A is a reflection matrix, that is A is of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where $a^2 + b^2 = 1$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

$\because A$ is symmetric $\Rightarrow b = c$ and $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

A is orthogonal $\Rightarrow \begin{cases} a^2 + b^2 = 1 \\ b^2 + d^2 = 1 \end{cases} \Rightarrow a^2 - d^2 = 0 \Rightarrow d = \pm a$

So $A = \begin{pmatrix} a & b \\ b & \pm a \end{pmatrix}$ where $a^2 + b^2 = 1$.

And $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} b \\ \pm a \end{pmatrix} = 0$. If $+$, then $2ab = 0 \Rightarrow$ either $a = 0$ ($\Leftrightarrow a = -a$) or $b = 0$ ($\Rightarrow a = \pm 1$)

$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\pm I_2$

If $-$, then $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = ab - ba = 0$

So we're done.

$\Rightarrow A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

4. Let k be a real number. Find an orthonormal basis of eigenvectors of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$, where A is the 3×3 matrix defined in terms of k by

$$A = \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$$

Note $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a(a^2-1) - (a-1) + (1-a)$

$= (a-1)[a(a+1) - 1 + (-1)]$

$= (a-1)(a^2+a-2)$

$= (a-1)^2(a+2)$

So $f_A(\lambda) = \begin{vmatrix} k-\lambda & 1 & 1 \\ 1 & k-\lambda & 1 \\ 1 & 1 & k-\lambda \end{vmatrix} = (k-\lambda-1)^2(k-\lambda+2)$

$\Rightarrow$ the eigenvalues of A are $k-1$ (AM $=2$) and $k+2$ (AM $=1$).

- $\underline{\underline{\lambda = k-1}}$ $\quad A-(k-1)I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_{k-1}$ has basis $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Gram-Schmidt: $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{v}_2^{\perp} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\Bigg(\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{=1}\Bigg)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

- $\underline{\underline{\lambda = k+2}}$ $\quad A-(k+2)I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

Observe that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the kernel $\quad -2+1+1=0$

$\Rightarrow \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

So $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an orthonormal eigenbasis of A. (Curiously, the eigenbasis is independent of k!)