1. Find a basis for the orthogonal complement of the subspace of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix}1\\-2\\0\\1\\-1\end{pmatrix}, \quad \begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}-2\\4\\1\\2\\3\end{pmatrix}$$

Let A be the matrix with these vectors as its columns.
We find a basis for $\ker(A^T) = \operatorname{im}(A)^T = \operatorname{span}\{\ldots\}^T$.

$$A^T = \begin{pmatrix}1 & -2 & 0 & 1 & -1\\ 1 & -2 & 1 & 0 & 0\\ -2 & 4 & 1 & 2 & 3\end{pmatrix} \xrightarrow[R_3+2R_1]{R_2-R_1} \begin{pmatrix}1 & -2 & 0 & 1 & -1\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 1 & 4 & 1\end{pmatrix} \xrightarrow{R_3-R_2}$$

$$\begin{pmatrix}1 & -2 & 0 & 1 & -1\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 5 & 0\end{pmatrix} \xrightarrow[\text{clear col 4}]{\frac{1}{5}R_3} \begin{pmatrix}1 & -2 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

$x_2 = s$, $x_5 = t$ free
$x_1 = 2s + t$
$x_3 = -t$
$x_4 = 0$

$$\Rightarrow \ker(A^T) \text{ has basis } \begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\\1\end{pmatrix}$$

2. Find a least squares solution to the system $\begin{cases} x + y = 59 \\ x - y = 1 \\ x - 2y = -29 \end{cases}$.

$$A = \begin{pmatrix}1 & 1\\ 1 & -1\\ 1 & -2\end{pmatrix} \Rightarrow A^TA = \begin{pmatrix}1 & 1 & 1\\ 1 & -1 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\\ 1 & -2\end{pmatrix} = \begin{pmatrix}3 & -2\\ -2 & 6\end{pmatrix}$$

note its rank is $2 = \#$cols $\Rightarrow \ker A = \{\vec{0}\}$.

$$\Rightarrow (A^TA)^{-1} = \frac{1}{14}\begin{pmatrix}6 & 2\\ 2 & 3\end{pmatrix}$$

$$\Rightarrow \vec{x}^* = \frac{1}{14}\begin{pmatrix}6 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & -2\end{pmatrix}\begin{pmatrix}59\\ 1\\ -29\end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix}8 & 4 & 2\\ 5 & -1 & -4\end{pmatrix}\begin{pmatrix}59\\ 1\\ -29\end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix}472 + 4 - 58\\ 295 - 1 + 116\end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix}418\\ 410\end{pmatrix} = \frac{1}{7}\begin{pmatrix}209\\ 205\end{pmatrix} \approx \begin{pmatrix}29.9\\ 29.3\end{pmatrix}.$$

3. A herd of guinea pigs is released on a previously cavy-free island in the South Atlantic ocean. The island is free of predators and has an abundance of hay, so their population is believed to grow exponentially, but it is unknown exactly how fast. Scientists keep a close eye on the herd and count the number of guinea pigs on the island each month for three months. Their findings are summarised in the following table.

time (in months) since release	0	1	2	3
number of guinea pigs	6	25	95	382

Find a function f in a single variable t that approximates how many guinea pigs will be on the island at time t months after their release.

[Hint: we should have $\log f(t) = a + bt$ for some real numbers a and b.]

We find a least squares solution to the system

$$\begin{cases} a = \log 6 \\ a + b = \log 25 \\ a + 2b = \log 95 \\ a + 3b = \log 382 \end{cases} \rightsquigarrow A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \Rightarrow (A^T A)^{-1} = \frac{1}{2 \times 5} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$$

$$\Rightarrow (A^T A)^{-1} A^T = \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ -3 & 1 & 1 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a^* \\ b^* \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} \log 6 \\ \log 25 \\ \log 95 \\ \log 382 \end{pmatrix} \approx \begin{pmatrix} 1.81 \\ 1.38 \end{pmatrix}$$

$$\text{So } f(t) = e^{1.81 + 1.38t}$$

$$= e^{1.81} \cdot (e^{1.38})^t$$

$$\approx \underline{\underline{6.11 \times (3.97)^t}}$$

4. Find a line of best fit for the data $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$, where $a_i \neq a_j$ for some $i \neq j$.

Suppose $y = c + dx$ for some real c, d. We approximate c and d by least squares. We have:

$$\begin{cases} c + da_1 = b_1 \\ c + da_2 = b_2 \\ \vdots \\ c + da_n = b_n \end{cases} \rightsquigarrow A = \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Then $$A^TA = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} n & a_1+a_2+\cdots+a_n \\ a_1+a_2+\cdots+a_n & a_1^2+a_2^2+\cdots+a_n^2 \end{pmatrix}$$

Write $s_a = a_1+a_2+\cdots+a_n$ and $q_a = a_1^2+a_2^2+\cdots+a_n^2$.

$$(A^TA)^{-1} = \begin{pmatrix} n & s_a \\ s_a & q_a \end{pmatrix}^{-1} = \frac{1}{nq_a - s_a^2} \begin{pmatrix} q_a & -s_a \\ -s_a & n \end{pmatrix}$$

$$\Rightarrow (A^TA)^{-1}A^T = \frac{1}{nq_a - s_a^2} \begin{pmatrix} q_a & -s_a \\ -s_a & n \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

$$= \frac{1}{nq_a - s_a^2} \begin{pmatrix} q_a - a_1 s_a & q_a - a_2 s_a & \cdots & q_a - a_n s_a \\ -s_a + na_1 & -s_a + na_2 & \cdots & -s_a + na_n \end{pmatrix}$$

$$\Rightarrow \vec{x}^* = \frac{1}{nq_a - s_a^2} \begin{pmatrix} q_a - a_1 s_a & q_a - a_2 s_a & \cdots & q_a - a_n s_a \\ -s_a + na_1 & -s_a + na_2 & \cdots & -s_a + na_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \frac{1}{nq_a - s_a^2} \begin{pmatrix} s_b q_a - (\vec{a}\cdot\vec{b}) s_a \\ -s_a s_b + (\vec{a}\cdot\vec{b}) n \end{pmatrix}$$

$$\Rightarrow \begin{cases} c = \dfrac{q_a s_b - (\vec{a}\cdot\vec{b}) s_a}{nq_a - s_a^2} = \dfrac{(\sum a_i^2)(\sum b_i) - (\sum a_i b_i)(\sum a_i)}{n(\sum a_i^2) - (\sum a_i)^2} \\[2ex] d = \dfrac{n(\vec{a}\cdot\vec{b}) - s_a s_b}{nq_a - s_a^2} = \dfrac{n(\sum a_i b_i) - (\sum a_i)(\sum b_i)}{n(\sum a_i^2) - (\sum a_i)^2} \end{cases}$$