1. (a) Show that the matrix $\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A}$ is orthogonal.

$$A^TA = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ A is orthogonal.

(b) Suppose that $a^2 + b^2 = 1$. Show that the matrix $\underbrace{\begin{pmatrix} a & b \\ b & -a \end{pmatrix}}_{B}$ is orthogonal.

$$B^TB = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ab-ba \\ ba-ab & b^2+a^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ B is orthogonal.

(c) (Try at home:) Show that all orthogonal 2×2 matrices are of the form (a) or (b). Hence all orthogonal transformations $\mathbb{R}^2 \to \mathbb{R}^2$ are rotations and reflections.

2. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let A be an orthogonal matrix. Then $\det(A) = \pm 1$.

<u>Always</u> If A is orthogonal then $A^T A = 1$. So

$$\underbrace{\det(A^T A)}_{\det(I_n) = 1} = \det(A^T)\det(A) = \det(A^2) = \det(A)^2$$

$$\Rightarrow \det(A) = \pm 1$$

(b) Let A be a 2×3 matrix. Then $A^T A$ is orthogonal.

<u>Never</u> $A^T A$ is a 3×3 matrix but

$$\operatorname{rank}(A^T A) = \dim(\operatorname{im} A^T A) \leq \dim(\operatorname{im} A^T) = \operatorname{rank}(A^T) \leq 2 < 3$$

since A^T is a 3×2 matrix

$\Rightarrow A^T A$ is not invertible $\Rightarrow$ not orthogonal.

(c) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If the angle between vectors $\vec{x}$ and $\vec{y}$ is θ, then the angle between $T(\vec{x})$ and $T(\vec{y})$ is θ.

<u>Always</u> Let φ be the angle between $T(\vec{x})$ and $T(\vec{y})$.

Then

$$\cos\varphi = \frac{T(\vec{x}) \cdot T(\vec{y})}{\|T(\vec{x})\| \, \|T(\vec{y})\|} \underset{\text{by orthogonality}}{=} \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} = \cos\theta$$

$\Rightarrow \varphi = \theta$ since $0 \leq \varphi, \theta \leq \pi$.

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(d) Suppose $\det(A) = 1$. Then A is orthogonal.

Sometimes

It's true when $A = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ $\det A = 1$, $A^T A = I_2^2 = I_2$

It's false when $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\det A = 1$

but $\left\| A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{2} \neq 1 = \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|$

$\Rightarrow$ A doesn't preserve lengths $\Rightarrow$ A is not orthogonal.

(e) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then T is orthogonal.

Sometimes

It's true if $\underbrace{T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{so } T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ z \\ y \end{pmatrix}}$, since $\underset{= T(\vec{e}_1)}{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}, \underset{= T(\vec{e}_2)}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}, \underset{= T(\vec{e}_3)}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$ are orthonormal

It's false if $\underbrace{T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{\text{so } T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ 0 \\ y+2z \end{pmatrix}}$ since $\left\| T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\| = 2 \neq 1 = \left\| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\|$

$\Rightarrow$ T doesn't preserve lengths.

3. (a) Find an orthonormal basis of the plane V in $\mathbb{R}^3$ described by the equation $2x+y-3z=0$.

Basis of V: $\underbrace{\begin{pmatrix}1\\-2\\0\end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{\vec{v}_2}$

Gram-Schmidt

- $\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\\0\end{pmatrix}$

- $\vec{v}_2^{\perp} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{5}\Bigg(\overbrace{\begin{pmatrix}1\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}^{=-1}\Bigg)\begin{pmatrix}1\\-2\\0\end{pmatrix}$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{5}\begin{pmatrix}1\\-2\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}6\\3\\5\end{pmatrix}$$

$$\Rightarrow \vec{u}_2 = \frac{1}{\sqrt{36+9+25}}\begin{pmatrix}6\\3\\5\end{pmatrix} = \frac{1}{\sqrt{70}}\begin{pmatrix}6\\3\\5\end{pmatrix}$$

So an orthonormal basis of V is

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\\0\end{pmatrix}, \frac{1}{\sqrt{70}}\begin{pmatrix}6\\3\\5\end{pmatrix}$$

(b) Find the matrix of orthogonal projection onto V.

Let $Q = \begin{pmatrix}1/\sqrt{5} & 6/\sqrt{70}\\ -2/\sqrt{5} & 3/\sqrt{70}\\ 0 & 5/\sqrt{70}\end{pmatrix}$. Then the matrix of orthogonal projection onto V is QQ^T:

$$\begin{pmatrix}1/\sqrt{5} & 6/\sqrt{70}\\ -2/\sqrt{5} & 3/\sqrt{70}\\ 0 & 5/\sqrt{70}\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5} & 0\\ 6/\sqrt{70} & 3/\sqrt{70} & 5/\sqrt{70}\end{pmatrix} = \begin{pmatrix}\frac{1}{5}+\frac{6}{70} & -\frac{2}{5}+\frac{18}{70} & 0+\frac{30}{70}\\ -\frac{2}{5}+\frac{18}{70} & \frac{4}{5}+\frac{9}{70} & 0+\frac{15}{70}\\ 0+\frac{30}{70} & 0+\frac{15}{70} & 0+\frac{25}{70}\end{pmatrix}$$

$$= \frac{1}{70}\begin{pmatrix}14+6 & -28+18 & 30\\ -28+18 & 56+9 & 15\\ 30 & 15 & 25\end{pmatrix} = \frac{1}{70}\begin{pmatrix}20 & -10 & 30\\ -10 & 65 & 15\\ 30 & 15 & 25\end{pmatrix} = \frac{1}{5}\begin{pmatrix}4 & -2 & 6\\ -2 & 13 & 3\\ 6 & 3 & 5\end{pmatrix}$$