**1.** Use the Gram–Schmidt process to turn the following sets of vectors into orthonormal sets spanning the same subspace.

(a) $\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_1}$

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}$$

(b) $\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2}$

- $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}$

- $$\begin{aligned}\vec{v}_2^{\perp} &= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \cdot 1 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\end{aligned}$$

$$\Rightarrow \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\frac{1}{2}\sqrt{6}} \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \underline{\underline{\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}}$$

(c) $\underbrace{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}_{\vec{v}_3}$

Like in (a) and (b),

- $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}}}$

- $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \frac{1}{2}\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$

  $\Rightarrow \vec{u}_2 = \underline{\underline{\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\\0\end{pmatrix}}}$

These are the same vectors as before, just with a 0 in their $4^{th}$ component.

- $$\begin{aligned}
\vec{v}_3^{\perp} &= \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 \\
&= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{1}{2}\Bigg(\underbrace{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}_{=1}\Bigg)\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} - \frac{1}{6}\Bigg(\underbrace{\begin{pmatrix}1\\2\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix}}_{=1}\Bigg)\begin{pmatrix}1\\2\\1\\0\end{pmatrix} \\
&= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \begin{pmatrix}1/2\\0\\-1/2\\0\end{pmatrix} - \begin{pmatrix}1/6\\2/6\\1/6\\0\end{pmatrix} \\
&= \frac{1}{6}\left[\begin{pmatrix}6\\0\\0\\6\end{pmatrix} - \begin{pmatrix}3\\0\\-3\\0\end{pmatrix} - \begin{pmatrix}1\\2\\1\\0\end{pmatrix}\right] \\
&= \frac{1}{6}\begin{pmatrix}2\\-2\\2\\6\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-1\\1\\2\end{pmatrix}
\end{aligned}$$

$$\Rightarrow \vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3^{\perp} = \frac{1}{\frac{1}{3}\sqrt{7}} \cdot \frac{1}{3}\begin{pmatrix}1\\-1\\1\\2\end{pmatrix} = \underline{\underline{\frac{1}{\sqrt{7}}\begin{pmatrix}1\\-1\\1\\2\end{pmatrix}}}$$

**2.** Find an orthonormal basis of the plane $x - 2y + z = 0$.

By inspection, two LI vectors in the plane are
$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Gram-Schmidt:

- $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

- $$\begin{aligned}
\vec{v}_2^{\perp} &= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 \\
&= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{5}\left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{5}\left[\begin{pmatrix} 0 \\ 5 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right] \\
&= \frac{1}{5}\begin{pmatrix} -2 \\ 4 \\ 10 \end{pmatrix} \\
&= \frac{2}{5}\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}
\end{aligned}$$

$$\Rightarrow \vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\frac{2}{5}\sqrt{30}} \cdot \frac{2}{5}\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$

So $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{30}}\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ is an orthonormal basis of the plane $x - 2y + z = 0$.

**3.** Define a matrix $A$ by

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\uparrow R_1 - 2R_2$$

(a) Find the rank of $A$.

$$\text{rref: } \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 \end{pmatrix} \xrightarrow[R_4 - 2R_1]{R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \xrightarrow{\text{evident}} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\underline{rk(A) = 2}}$$

(b) Find an orthonormal basis of the image of $A$.

$$\dim(\operatorname{im} A) = rk(A) = 2 \Rightarrow \operatorname{im} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

$$\underline{\text{Gram-Schmidt}}: \quad \vec{u}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{v}_2^{\perp} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} - \underbrace{\frac{1}{6}(2+0+0+6)}_{=4/3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{3}\left[ \begin{pmatrix} 6 \\ 0 \\ 3 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 0 \\ 8 \end{pmatrix} \right] = \frac{1}{3} \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{u}_2 = \frac{1}{\sqrt{4+16+9+1}} \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}$$

So $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ -4 \\ 3 \\ 1 \end{pmatrix}$ is an orthonormal basis of $\operatorname{im}(A)$.

(c) Find an orthonormal basis of the kernel of $A$.

From rref above, we see $A\vec{x} = \vec{0} \Leftrightarrow x_3 = s,\ x_4 = t,\ x_2 = -s-t$,
$x_1 = s + 2t$

$$\Rightarrow \ker A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\underline{\text{Gram-Schmidt}}: \quad \vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_2^{\perp} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \underbrace{\frac{1}{3}(2+1+0+0)}_{=1} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

So $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ is an orthonormal basis of $\ker A$.