Math 290-2 Class 2

Wednesday 9th January 2019

Ortho-more-mal

Recall that vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ in $\mathbb{R}^n$ are orthonormal if and only if

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and that the **orthogonal projection** of a vector $\vec{x}$ onto a subspace V with orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ is given by

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_k \cdot \vec{x})\vec{u}_k$$

Consequently, if $\mathfrak{B} = \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is an orthonormal basis of $\mathbb{R}^n$, then $\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$.

This makes computing coordinate vectors with respect to orthonormal bases extremely easy:

$$[\vec{x}]_{\mathfrak{B}} = \begin{pmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \\ \vdots \\ \vec{u}_n \cdot \vec{x} \end{pmatrix}$$

The **orthogonal complement** of a subspace V of $\mathbb{R}^n$ is the subspace $V^\perp$ of $\mathbb{R}^n$ consisting of all vectors perpendicular to those in V:

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0 \text{ for all } \vec{v} \text{ in } V\} = \ker(\text{proj}_V)$$

Note that $\dim(V) + \dim(V^\perp) = n$ by the rank–nullity theorem, since $V = \text{im}(\text{proj}_V)$.

Some geometry

Dot products, lengths and angles all neatly related by the following theorem: if $\vec{x}$ and $\vec{y}$ are any two vectors in $\mathbb{R}^n$, such that the angle between $\vec{x}$ and $\vec{y}$ is θ (where $0 \leqslant \theta \leqslant \pi$), then

$$\vec{x} \cdot \vec{y} = ||\vec{x}||\,||\vec{y}|| \cos\theta$$

Some more fun facts:

- $|\vec{x} \cdot \vec{y}| \leqslant ||\vec{x}||\,||\vec{y}||$ — this is called the *Cauchy–Schwarz inequality*;
- $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ if and only if $\vec{x}$ and $\vec{y}$ are orthogonal.

1. (a) Verify that $\mathfrak{B} = \begin{pmatrix} 1/2 \\ 0 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an orthonormal basis of $\mathbb{R}^3$.

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ \vec{u} & \vec{v} & \vec{w} \end{array}$$

$$\vec{u}\cdot\vec{u} = \vec{v}\cdot\vec{v} = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$\vec{w}\cdot\vec{w} = 0^2 + 1^2 + 0^2 = 1$$

$$\vec{u}\cdot\vec{v} = \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0$$

$$\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} = *\cdot 0 + 0\cdot 1 + *\cdot 0 = 0$$

$\vec{u}, \vec{v}, \vec{w}$ are orthonormal

& $\dim(\mathbb{R}^3) = 3$

$\Rightarrow$ $\vec{u}, \vec{v}, \vec{w}$ are an orthonormal basis of $\mathbb{R}^3$

(b) Find the coordinates of $\vec{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ with respect to $\mathfrak{B}$.

$$\vec{a}\cdot\vec{u} = -\frac{1}{2} + 0 + 0 = -\frac{1}{2}$$

$$\vec{a}\cdot\vec{v} = -\frac{\sqrt{3}}{2} + 0 + 0 = -\frac{\sqrt{3}}{2}$$

$$\vec{a}\cdot\vec{w} = 0 + 1 + 0 = 1$$

$$\Rightarrow \quad [\vec{a}]_{\mathfrak{B}} = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ 1 \end{pmatrix}.$$

2. Let V be the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(a) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto V;

$\left\|\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}\right\| = \sqrt{8} = 2\sqrt{2} \Rightarrow \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a unit vector parallel to $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$\left\|\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\| = \sqrt{6} \Rightarrow \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is a unit vector parallel to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

& $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2 + 0 - 2 = 0$

$\Rightarrow \vec{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ & $\vec{v} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are orthonormal

$\Rightarrow \mathcal{B} = \vec{u}, \vec{v}$ is an orthonormal basis for V.

$$\Rightarrow \mathrm{proj}_V\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left(\vec{u} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)\vec{u} + \left(\vec{v} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)\vec{v}$$

$$= \frac{1}{2}\underbrace{\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)}_{=2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{6}\underbrace{\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)}_{=2}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

(b) Find the orthogonal complement of V.

Since $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is a basis of V, a vector $\vec{x}$ is perpendicular to $V \Leftrightarrow \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \vec{x} = 0$

Note $\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 0x_2 + 2x_3 \\ x_1 + 2x_2 - x_3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cdot \vec{x} \\ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \vec{x} \end{pmatrix}$

So $V^\perp = \ker\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$x_3 = t$ free, $x_1 = -t$, $x_2 = t$

So $V^\perp = \mathrm{span}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

3. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors in $\mathbb{R}^3$ defined by

$$\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

(a) Show that $\vec{c}$ is in the orthogonal complement of $\operatorname{span}\{\vec{a}, \vec{b}\}$.

$\vec{c}$ is in $\operatorname{span}\{\vec{a}, \vec{b}\}^\perp \iff \vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b}$

$$\vec{a} \cdot \vec{c} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0 - 2 + 2 = 0 \Rightarrow \vec{a} \perp \vec{c}$$

$$\vec{b} \cdot \vec{c} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0 - 2 + 2 = 0 \Rightarrow \vec{b} \perp \vec{c}$$

So $\vec{c}$ is in $\operatorname{span}\{\vec{a}, \vec{b}\}^\perp$

(b) Find the angle between $\vec{a}$ and $\vec{b}$.

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\| \left\|\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\right\|}$$

$$= \frac{-1 + 1 + 4}{\sqrt{1+1+4}\sqrt{1+1+4}}$$

$$= \frac{4}{6} = \frac{2}{3}$$

$$\Rightarrow \theta = \arccos\left(\frac{2}{3}\right) //$$

4. For each of the following (true) statements, explain why it is true.

(a) If $|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\|\vec{y}\|$, then $\vec{x}$ and $\vec{y}$ are parallel.

Trivial case: if $\vec{x}$ or $\vec{y} = \vec{0}$ then this is immediate.

Otherwise, angle between $\vec{x}$ and $\vec{y}$, $0 \leq \theta \leq \pi$

$$\|\vec{x}\|\|\vec{y}\| = |\vec{x}\cdot\vec{y}| = \|\vec{x}\|\|\vec{y}\| |\cos\theta| \Rightarrow |\cos\theta| = 1$$

$$\Rightarrow \cos\theta = \pm 1 \quad \Rightarrow \theta = 0 \text{ or } \pi$$

$\Rightarrow \vec{x}$ and $\vec{y}$ are parallel

(b) Let ℓ be a line in $\mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be nonzero vectors in $\mathbb{R}^n$. If $\vec{v}$ is parallel to ℓ, and the equation $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ holds, then $\vec{w}$ is in $\ell^\perp$.

Since $\ell = \text{span}\{\vec{v}\}$, $\vec{w}$ is in $\ell^\perp \Leftrightarrow \vec{w} \perp \vec{v}$.

But if $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$, then $\vec{v} \perp \vec{w}$

$\Rightarrow \vec{w}$ is in $\ell^\perp$.

(c) Let V be a subspace of $\mathbb{R}^n$ and let $\vec{x}$ be a vector in $\mathbb{R}^n$. Then $\|\text{proj}_V(\vec{x})\| \leqslant \|\vec{x}\|$.

We can write $\vec{x}$ as $\vec{x}^{\parallel} + \vec{x}^{\perp}$ where

- $\vec{x}^{\parallel} = \text{proj}_V(\vec{x})$ is in V
- $\vec{x}^{\perp}$ is perpendicular to V $(\Rightarrow \vec{x}^{\perp} \perp \vec{x}^{\parallel})$

So
$$\|\vec{x}\|^2 = \|\vec{x}^{\parallel}\|^2 + \|x^{\perp}\|^2$$
$$\geqslant \|\vec{x}^{\parallel}\|^2 = \|\text{proj}_V(\vec{x})\|^2$$

and $\|\text{proj}_V(\vec{x})\|^2 \leq \|\vec{x}\|^2 \Rightarrow \|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$

since both quantities are $\geqslant 0$.