1. Perform the following tedious algebraic tasks.

(a) Find the roots of the polynomial $f(x) = x^2 + 4x + 8$.

$$x^2 + 4x + 8 = (x+2)^2 - 4 + 8$$
$$= (x+2)^2 + 4$$
$$= 0 \iff (x+2)^2 = -4$$
$$\iff x + 2 = \pm 2i$$
$$\iff \underline{\underline{x = -2 \pm 2i}}$$

(b) Find the square roots of the complex number $9i$.

We solve $x^2 = 9i$. So let $x = a + bi$. Then

$$(a+bi)^2 = (a^2 - b^2) + 2abi = 9i$$

$$\Rightarrow \begin{cases} a^2 - b^2 = 0 \longrightarrow b = \pm a \\ 2ab = 9 \longrightarrow \pm 2a^2 = 9 \end{cases}$$

Since a & b are real, we can't have $-2a^2 = 9$ so $b \neq -a$. So $b = a$, and $2a^2 = 9 \Rightarrow a = \frac{\pm 3}{\sqrt{2}} = \frac{\pm 3}{2}\sqrt{2}$

$$\Rightarrow \quad x = \pm \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i \right)$$

(c) Let $z = a + bi$ be a complex number. Find $z + \bar{z}$ and $z\bar{z}$, where $\bar{z} = a - bi$ is the complex conjugate of z, and observe that $z + \bar{z}$ and $z\bar{z}$ are both real.

$$z + \bar{z} = (a + bi) + (a - bi) = \underline{\underline{2a}}$$

$$z\bar{z} = (a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2$$
$$= \underline{\underline{a^2 + b^2}}$$

These are indeed both real! Wow!

$$\left[\begin{array}{l} \frac{z+\bar{z}}{2} \text{ is the real part of } z. \\ |z| = \sqrt{z\bar{z}} \text{ is the modulus of } z. \end{array} \right]$$

2. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$f_A(\lambda) = \lambda^2 - 4\lambda + 5 = (\lambda-2)^2 - 4 + 5$$
$$= (\lambda-2)^2 + 1$$
$$= 0 \iff (\lambda-2)^2 = -1$$
$$\iff \lambda - 2 = \pm i$$
$$\iff \lambda = 2 \pm i$$

$$A - (2+i)I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \Rightarrow E_{2+i} = \text{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$$

$$A - (2-i)I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \Rightarrow E_{2-i} = \text{span}\left\{\begin{pmatrix} i \\ -1 \end{pmatrix}\right\}$$

3. Show that if an 5×5 matrix A has characteristic polynomial

$$f_A(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4 - \lambda^5$$

then $\det(A) = a_0$ and $\text{tr}(A) = a_4$.

By the fundamental theorem of algebra there exist complex numbers $\lambda_1, \lambda_2 \dots \lambda_n$ such that

$$\left.\begin{array}{l} \\ f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \end{array}\right\} \text{in this case, } n = 5$$

$$\Rightarrow a_0 + \cdots + a_4\lambda^4 - \lambda^5$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)(\lambda_4 - \lambda)(\lambda_5 - \lambda)$$
$$= \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 + (\text{junk})\lambda + (\text{junk})\lambda^2 + (\text{junk})\lambda^3$$
$$+ (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)\lambda^4 - \lambda^5$$

$$\Rightarrow \begin{cases} a_0 = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = \det A \\ a_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = \text{tr}\, A \end{cases}$$

4. For each of the following statements, determine if it is always, sometimes or never true.

(a) Let A be a 2×2 real matrix. Then A can be diagonalised, provided the diagonal entries are allowed to be complex numbers.

Sometimes! Complex numbers have nothing to do with it.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal(isable)

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonal(isable), as we have proved at least twice already!

(b) Let B be a 3×3 real matrix. Then B has exactly one non-real eigenvalue.

Never! Since B is real, $f_B(\lambda)$ is a cubic polynomial with real coefficients $\Rightarrow$ it has at least one real root λ_1. So

$$f_B(\lambda) = (\lambda_1 - \lambda)(a + b\lambda + \lambda^2) \text{ for some real } a, b$$

If $b^2 - 4a$ is $\begin{Bmatrix} \geq 0 \\ < 0 \end{Bmatrix}$ then f_B has $\begin{Bmatrix} \text{no non-real roots} \\ \text{2 non-real roots} \end{Bmatrix}$.

(c) Let C be an $n \times n$ real matrix and let λ be an eigenvalue of C. If every vector in E_λ is real, then λ is real.

Always!

If $\vec{v}$ is real & λ is non-real, then:

By "A", I mean "C"

- $A\vec{v}$ is real
- $\lambda\vec{v}$ has a non-real entry if $\vec{v} \neq \vec{0}$

So the only vector $\vec{v}$ s.t. $A\vec{v} = \lambda\vec{v}$ is $\vec{v} = \vec{0}$

So λ is not an eigenvalue of A.

So the only way E_λ can consist of only real vectors is if λ is real.