1. Diagonalise the following matrix.

$$A = \begin{pmatrix} -2 & 3 & 3 \\ 0 & -5 & -3 \\ 0 & 6 & 4 \end{pmatrix}$$

$$f_A(\lambda) = \begin{vmatrix} -2-\lambda & 3 & 3 \\ 0 & -5-\lambda & -3 \\ 0 & 6 & 4-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 6 & 4-\lambda \end{vmatrix}$$

$$= (-2-\lambda)((-5-\lambda)(4-\lambda)+18)$$
$$= (-2-\lambda)(-20+\lambda+\lambda^2+18)$$
$$= (-2-\lambda)(\lambda^2+\lambda-2)$$
$$= (-2-\lambda)(\lambda+2)(\lambda-1)$$
$$= -(\lambda+2)^2(\lambda-1)$$

The eigenvalues of A are -2 (w. alg. mult. $= 2$) and 1 (w. alg. mult. $= 1$)

$\underline{\underline{\lambda = -2}}$ $\quad A + 2I = \begin{pmatrix} 0 & 3 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

x_1, x_3 free & $x_2 = -x_3$

so vectors in the kernel take the form $\begin{pmatrix} s \\ -t \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow E_{-2} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$

$\underline{\underline{\lambda = 1}}$ $\quad A - I = \begin{pmatrix} -3 & 3 & 3 \\ 0 & -6 & -3 \\ 0 & 6 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -6 & 6 & 6 \\ 0 & -6 & -3 \\ 0 & 6 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -6 & 0 & 3 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{almost rref}} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

x_3 free, $2x_1 = x_3$, $2x_2 = -x_3$

so vectors in the kernel take the form $\begin{pmatrix} s \\ -s \\ 2s \end{pmatrix} = s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\}$

So $S^{-1}AS = \underline{\underline{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}$ where $S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

2. Compute $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^{1000}$

First we diagonalise $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$.

$tr(A) = 3, \quad \det(A) = -4 + 6 = 2$

$\Rightarrow f_A(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$

$\underline{\underline{\lambda = 2}}$: $A - 2I = \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}$

$\Rightarrow \ker(A - 2I) = \text{span}\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$

$\underline{\underline{\lambda = 1}}$: $A - I = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}$

$\Rightarrow \ker(A - I) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

So $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}^{-1}}$

$= \frac{1}{3 - 2} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 1^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & -2^n \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 2^n - 2 & -3 \cdot 2^n + 3 \\ 2 \cdot 2^n - 2 & -2 \cdot 2^n + 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 2^n - 2 & -3 \cdot 2^n + 3 \\ 2^{n+1} - 2 & -2^{n+1} + 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}^{1000} = \begin{pmatrix} 3 \cdot 2^{1000} - 2 & -3 \cdot 2^{1000} + 3 \\ 2^{1001} - 2 & -2^{1001} + 3 \end{pmatrix}$$

3. For each of the following statements, determine whether it is always, sometimes or never true.

(a) An $n \times n$ matrix A with n distinct eigenvalues is diagonalisable.

Always. Each eigenvalue has geometric multiplicity ≥ 1.
If there are n eigenvalues then their geometric multiplicities add up to n

$\Rightarrow$ the eigenvectors of A form a basis of $\mathbb{R}^n$, w.r.t. which the matrix of A is diagonal

$\Rightarrow$ A is diagonalisable.

(b) Let A be a 3×3 diagonalisable matrix. Then $f_A(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1$.

Never (... if we mean "diagonalisable over $\mathbb{R}$")

$\lambda^3 + \lambda^2 + \lambda + 1 = (\lambda + 1)(\lambda^2 + 1)$ & $\lambda^2 + 1$ has no real roots

The only real eigenvalue of A is -1 and its geometric multiplicity is 1, so the (real) eigenvectors of A do not form a basis of $\mathbb{R}^n$

$\Rightarrow$ A is not diagonalisable.

(c) Let A be a non-diagonalisable $n \times n$ matrix. Then A^2 is not diagonalisable.

Sometimes.

- True when $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
  The only eigenvalue of both A & A^2 is 1
  $A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\quad A - 2I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$
  Both have rank 1 $\Rightarrow$ geo. mul. of $1 = 1 < 2 =$ alg. mul.
  $\Rightarrow$ neither A nor A^2 is diagonalisable.
- False when $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
  The only eigenvalue of A is 0 & $A = A - 0I$ has rank 1 $\Rightarrow$ geo. mul. of $0 = 1 < 2 =$ ~~geo~~ alg. mul.
  $\Rightarrow$ A is not diagonalisable.
  But A^2 is diagonal so is certainly diagonalisable.