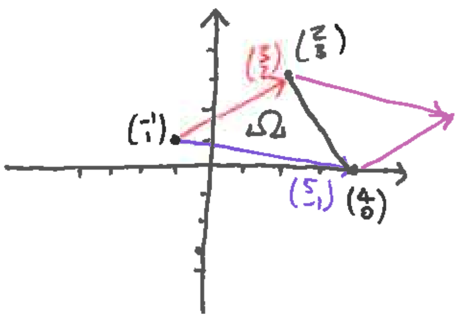
1. Let Ω be the triangle in $\mathbb{R}^2$ whose vertices have position vectors $\begin{pmatrix}-1\\1\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\0\end{pmatrix}$.

Find the area of Ω.



area of triangle

$$= \frac{1}{2}(\text{area of parallelogram})$$

$$= \frac{1}{2}\left|\det\begin{pmatrix}3 & 5\\ 2 & -1\end{pmatrix}\right|$$

$$= \frac{1}{2}\left|-3-10\right|$$

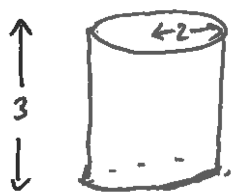
$$= \frac{1}{2}\left|-13\right|$$

$$= \underline{\underline{\frac{13}{2}}}$$

2. Let T be the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\vec{x}) = \begin{pmatrix} 6 & -1 & 1 \\ -3 & 1 & 2 \\ 4 & -1 & -3 \end{pmatrix} \vec{x}$$

Given that a region Ω of $\mathbb{R}^3$ is transformed via T to a cylinder of height 3 and radius 2, find the volume of Ω.



(not to scale...)

volume of cylinder $= \pi \cdot 2^2 \cdot 3 = \underline{\underline{12\pi}}$

$\Rightarrow$ volume of Ω

$$= \frac{\text{volume of } T(\Omega)^{\leftarrow = 12\pi}}{\left|\det\begin{pmatrix} 6 & -1 & 1 \\ -3 & 1 & 2 \\ 4 & -1 & -3 \end{pmatrix}\right|}$$

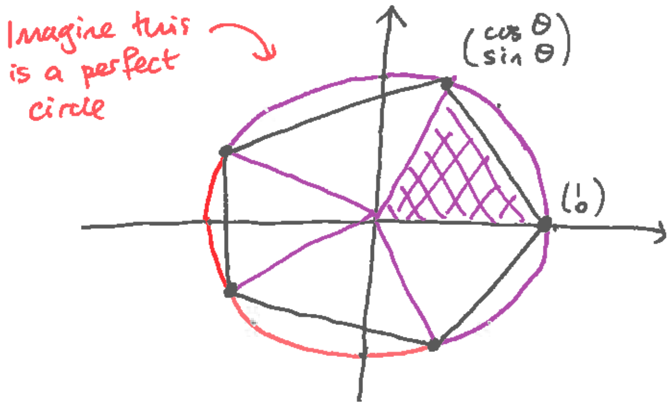
add row 2 to rows 1 and 3

$$\text{Now } \begin{vmatrix} 6 & -1 & 1 \\ -3 & 1 & 2 \\ 4 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 3 \\ -3 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 1 & -1 \end{vmatrix} = -3-3 = -6$$

expand down 2nd column

$$\Rightarrow \text{volume of } \Omega = \frac{12\pi}{|-6|} = \frac{12\pi}{6} = \underline{\underline{2\pi}}$$

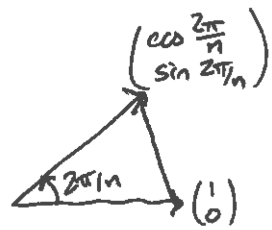
3. Find the area of a regular n-gon inscribed in a circle of radius 1.



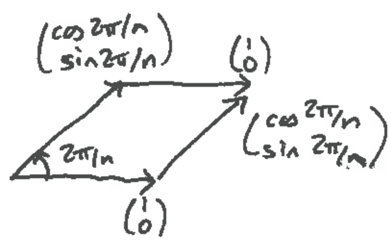
$$\theta = \frac{2\pi}{n}$$

area of n-gon

$= n \times$ area of



$= \frac{n}{2} \times$ area of



$$= \frac{n}{2} \left| \det \begin{pmatrix} 1 & \cos 2\pi/n \\ 0 & \sin 2\pi/n \end{pmatrix} \right|$$

$$= \frac{n}{2} \left| \sin \frac{2\pi}{n} \right|$$

since $\sin\theta > 0$ when $0 < \theta < \pi$ (and $n > 2$ since we're talking about an n-gon!)

$$= \underline{\underline{\frac{n}{2} \sin \frac{2\pi}{n}}}$$

4. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let A be a 3×3 matrix and let $\Omega \subseteq \mathbb{R}^3$ be a two-dimensional region with area $k \geqslant 0$. Then the area of $T(\Omega)$ is $k \cdot |\det(A)|$.

Sometimes

If $T(\vec{x}) = \vec{x}$ then $T(\Omega) = \Omega$ & $\det(A) = 1$

$\Rightarrow$ area of Ω = (area of Ω)$\cdot |\det A|$

If $T(\vec{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \text{projection onto} \\ (y,z)\text{-plane} \end{pmatrix}$ then $\det(A) = 0$

but $T(\Omega) = \Omega$ for any region of the (y,z)-plane,

e.g. $\Omega = \begin{Bmatrix} \text{unit circle} \\ \text{in } (y,z)\text{-plane} \end{Bmatrix} \Rightarrow \text{area}(\Omega) = \pi \neq 0 = \text{area}(T(\Omega))|\det A|$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation or reflection and let Ω be a region in $\mathbb{R}^3$. Then $T(\Omega)$ and Ω have the same volume.

Always

→ If T is a rotation, we saw last time that there is a basis $\mathcal{B}$ of $\mathbb{R}^3$ s.t. the $\mathcal{B}$-matrix of T is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, which has determinant 1 $\Rightarrow$ expansion factor $= 1$.

→ If T is a reflection, let $\vec{v}_1, \vec{v}_2$ be a basis of the plane of reflection & $\vec{v}_3$ be orthogonal to the plane. Then the matrix of T w.r.t. $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which has determinant $-1 \Rightarrow$ expansion factor $= 1$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and let $\vec{v}$ and $\vec{w}$ be nonzero vectors such that $T(\vec{v}) = \lambda\vec{v}$ and $T(\vec{w}) = \mu w$ for some scalars $\lambda \neq \mu$. Then the expansion factor of T is $|\lambda\mu|$.

Always

Let $\mathcal{B} = \vec{v}, \vec{w}$. Then $\mathcal{B}$ is a basis of $\mathbb{R}^2$ — if it weren't then $\vec{w}$ would be parallel to $\vec{v}$

$\Rightarrow T(\vec{w}) = T(k\vec{v}) = k\,T(\vec{v}) = k\lambda\vec{v} = \lambda k\vec{v} = \lambda\vec{w}$

but this is impossible since $\lambda \neq \mu$.

The $\mathcal{B}$-matrix of T is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Letting A be the standard matrix of T, we have $A = S\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}S^{-1}$ where S is the transition matrix of $\mathcal{B} \Rightarrow A$ is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

$\Rightarrow |\det A| = \left|\det\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right| = |\lambda\mu|$.