Math 290-1 Class 20 — review for midterm 2

Monday 12th November 2018

1. For each of the following statements, determine whether it is always, sometimes or never true.

(a) Let A be a 4×5 matrix with rank 4. There is a 4×3 matrix B and a 3×5 matrix C such that $A = BC$.

<u>Never</u> $\quad \mathrm{im}(A)$ is contained in $\mathrm{im}(B)$, since

$\vec{x}$ in $\mathrm{im}(A) \Rightarrow \vec{x} = A\vec{y} = BC\vec{y} = B(C\vec{y})$ for some $\vec{y}$

$\Rightarrow \vec{x} = B\vec{z}$ for some $\vec{z}$ (namely $\vec{z} = C\vec{y}$)

$\Rightarrow \vec{x}$ in $\mathrm{im}(B)$

But then $\underbrace{\mathrm{rank}(A)}_{=\,4} = \dim(\mathrm{im}\,A) \leq \dim(\mathrm{im}\,B) = \mathrm{rank}(B) \leq \#\text{cols of } B = 3$

$\Rightarrow 4 \leq 3$ — false!

(b) Let V be a subspace of $\mathbb{R}^3$. Then $V = \ker(T)$ for some linear transformation T.

<u>Always</u>

- If $V = \{\vec{0}\}$ then $V = \ker(I_3)$
- If V is a line, then $V = \ker\left(\begin{smallmatrix}\text{orthogonal projection}\\ \text{onto the plane perp. to } V\end{smallmatrix}\right)$
- If V is a plane, then $V = \ker\left(\begin{smallmatrix}\text{orthogonal projection}\\ \text{onto the line perp. to } V\end{smallmatrix}\right)$
- If $V = \mathbb{R}^3$, then $V = \ker(0)$ ← 3×3 zero matrix

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be distinct bases of $\mathbb{R}^2$. For every linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, the $\mathfrak{A}$-matrix of T is equal to the $\mathfrak{B}$-matrix of T.

<u>Sometimes</u>

- If $\mathcal{A} = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}}_{\text{transition mx} = I_2}$ and $\mathcal{B} = \underbrace{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}}_{\text{transition mx} = 2I_2}$, then

  for all 2×2 matrices A, $\quad I_2^{-1} A I_2 = A$

  and $(2I_2)^{-1} A (2I_2) = \frac{1}{2}\cdot I_2^{-1} \cdot A \cdot 2I_2 = A$

  $\to$ <u>true</u>

- If $\mathcal{A} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$

  and $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\2y\end{pmatrix}$ for all $\begin{pmatrix}x\\y\end{pmatrix}$ in $\mathbb{R}^2$ — since $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}$

  then $\mathcal{A}$-mx $= \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} \neq \mathcal{B}$-mx $= \begin{pmatrix}1 & -1\\0 & 2\end{pmatrix} \to$ <u>false</u>

2. For each of the following questions, determine whether it is true or false.

(a) Let $\vec{a}$ and $\vec{b}$ be vectors in $\mathbb{R}^n$. If $\vec{c}$ is in $\operatorname{span}(\vec{a},\vec{b})$, then $T(\vec{c})$ is in $\operatorname{span}(T(\vec{a}),T(\vec{b}))$.

True

If $\vec{c} = \lambda\vec{a} + \mu\vec{b}$

then $T(\vec{c}) = T(\lambda\vec{a} + \mu\vec{b}) = \lambda T(\vec{a}) + \mu T(\vec{b})$

by linearity of T.

$\Rightarrow T(\vec{c})$ is in $\operatorname{span}(T(\vec{a}), T(\vec{b}))$

(b) The matrices $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}$ are similar.

False. If they were similar then they'd have the same determinant. Expanding down 3rd column:

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & k \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2 + 2k$$

$$= \begin{cases} 2 & \text{if } k=2 \\ 4 & \text{if } k=3 \end{cases}$$

So $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} \neq \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} \Rightarrow$ not similar

(c) The transition matrix of a basis $\mathfrak{B}$ of $\mathbb{R}^n$ is invertible.

True. If $\mathcal{B} = \vec{v}_1, \dots, \vec{v}_n$ is a basis then

$\vec{v}_1, \dots, \vec{v}_n$ are LI

$\Rightarrow \begin{pmatrix} \vdots & & \vdots \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vdots & & \vdots \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ has a unique solution (namely $\vec{c} = \vec{0}$)

$\Rightarrow \operatorname{rank}\begin{pmatrix} \vdots & & \vdots \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vdots & & \vdots \end{pmatrix} = n$

$\Rightarrow \begin{pmatrix} \vdots & & \vdots \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vdots & & \vdots \end{pmatrix}$ is invertible $\because$ it's an $n \times n$ matrix with rank n.

3. Let k be some number. In terms of k, find the dimension of the image of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\vec{x}) = \underbrace{\begin{pmatrix} 1 & k & k^2 \\ k & 3k^2 & 2k^3 \\ 1 & 3k & 2k^2 \end{pmatrix}}_{:= A} \vec{x}$$

We compute rank(A), which is equal to dim(im T).

Row operations:

$$\begin{pmatrix} 1 & k & k^2 \\ k & 3k^2 & 2k^3 \\ 1 & 3k & 2k^2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & k & k^2 \\ 0 & 0 & 0 \\ 1 & 3k & 2k^2 \end{pmatrix} \quad (\mathrm{II}) - k(\mathrm{III})$$

$$\longrightarrow \begin{pmatrix} 1 & k & k^2 \\ 0 & 0 & 0 \\ 0 & 2k & k^2 \end{pmatrix} \quad (\mathrm{III}) - (\mathrm{I})$$

$$\longrightarrow \begin{pmatrix} 1 & k & k^2 \\ 0 & 2k & k^2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{swap } (\mathrm{II}), (\mathrm{III})$$

If $k \neq 0$ then rank$(A) = 2$ = dim(im T)

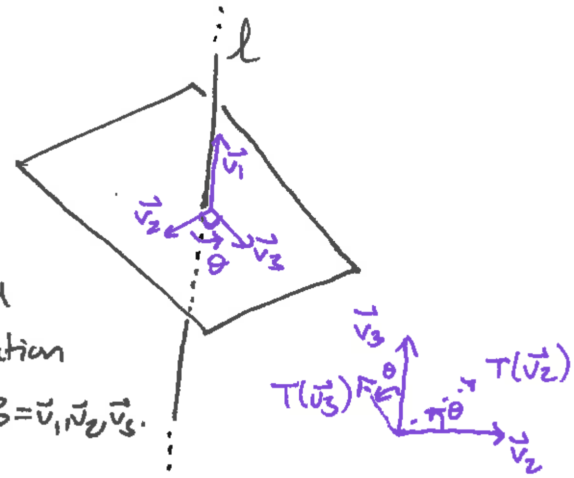
If $k = 0$ then rank$(A) = 1$ = dim(im T).

$0 \le \theta < 2\pi$ through the origin

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by rotation by an angle θ about a line ℓ. Give a geometric description of a basis of $\mathbb{R}^3$ with respect to which the matrix of T is of the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

and find the values of a, b, c and d in this case.



Let $\vec{v}_1$ be a nonzero vector parallel to ℓ, and let $\vec{v}_2, \vec{v}_3$ be a basis of the plane orthogonal to ℓ, oriented such that the rotation moves $\vec{v}_2$ towards $\vec{v}_3$; and let $\mathcal{B} = \vec{v}_1, \vec{v}_2, \vec{v}_3$. & with $\vec{v}_2 \perp \vec{v}_3$.

Then $T(\vec{v}_1) = \vec{v}_1$

$T(\vec{v}_2) = (\cos\theta)\vec{v}_2 + (\sin\theta)\vec{v}_3$

$T(\vec{v}_3) = (-\sin\theta)\vec{v}_2 + (\cos\theta)\vec{v}_3$

$$\Rightarrow \mathcal{B}\text{-mx of } T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

For which values of θ is this matrix diagonal?

The $\mathcal{B}$-mx is diagonal

$\Leftrightarrow (-\sin\theta = 0 \text{ and}) \sin\theta = 0$

$\Leftrightarrow \underline{\underline{\theta = 0}}$ or $\underline{\underline{\theta = \pi}}$