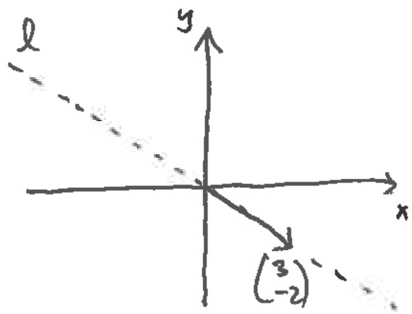
Math 290-1 Class 11 — review for midterm 1

Monday 22nd October 2018

1. Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects each vector through the line $2x+3y=0$.

[*You may use the fact that the reflection of a vector* $\vec{x}$ *through a line* ℓ *with direction* $\vec{a}$ *is given by the formula* $\mathrm{ref}_\ell(\vec{x}) = 2\left(\frac{\vec{x}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a} - \vec{x}$.]



$$\mathrm{ref}_\ell\begin{pmatrix}x\\y\end{pmatrix} = \frac{2\begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}3\\-2\end{pmatrix}}{\begin{pmatrix}3\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\end{pmatrix}}\begin{pmatrix}3\\-2\end{pmatrix} - \begin{pmatrix}x\\y\end{pmatrix}$$

$$= \frac{2(3x-2y)}{3^2+(-2)^2}\begin{pmatrix}3\\-2\end{pmatrix} - \begin{pmatrix}x\\y\end{pmatrix}$$

$$\Rightarrow \mathrm{ref}_\ell\begin{pmatrix}1\\0\end{pmatrix} = \frac{2\times3}{13}\begin{pmatrix}3\\-2\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}18/13\\-12/13\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}5/13\\-12/13\end{pmatrix}$$

$$\mathrm{ref}_\ell\begin{pmatrix}0\\1\end{pmatrix} = \frac{2\times(-2)}{13}\begin{pmatrix}3\\-2\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-12/13\\8/13\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-12/13\\-5/13\end{pmatrix}$$

So the matrix of ref_ℓ is $\begin{pmatrix}5/13 & -12/13\\-12/13 & -5/13\end{pmatrix}$.

Is T invertible? If not, explain why; if so, find the matrix of T^{-1}.

Yes — it is a reflection $\Rightarrow T^{-1} = T$

So the matrix of T^{-1} is $\begin{pmatrix}5/13 & -12/13\\-12/13 & -5/13\end{pmatrix}$

2. For each of the following statements about $n \times n$ matrices A, B and C, determine whether it is always true, sometimes true, or never true.

(a) If $AB = C$, then $B = CA^{-1}$.

Sometimes. If $A = B = C = I_2$ then $AB = C$ and $B = CA^{-1}$ since all four of AB, C, B and CA^{-1} equal I_2.

But $\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{C}$ and $\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{B} \neq \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{C}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A^{-1}=A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(b) For each vector $\vec{v}$ in $\mathbb{R}^n$, the vector $A\vec{v}$ is a linear combination of the columns of A.

Always. If $A = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

then $A\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n$.

(c) $\text{rank}(AB) = \text{rank}(A)\text{rank}(B)$

Sometimes. $\underline{\underline{0}} = \text{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \text{rank}\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] = \text{rank}\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]\text{rank}\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] = \underline{\underline{0 \times 0}}$

but $\underline{\underline{2}} = \text{rank}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{rank}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \neq \text{rank}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\text{rank}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \times 2 = \underline{\underline{4}}$

(d) If $\text{rank}(A) < n$, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.

Always. The system is consistent $\because$ $A\vec{0} = \vec{0}$ but it has at least one free variable since A is $n \times n$ and has rank $< n$.

(e) If ABC is invertible, then B is invertible.

Always. ABC invertible $\Rightarrow ABCD = I_n$ for some matrix D
$\Rightarrow A$ is invertible

and $DABC = I_n \Rightarrow C$ is invertible

$ABCD = I_n \Rightarrow B = A^{-1}D^{-1}C^{-1} \Rightarrow B$ is invertible as it is a product of invertible matrices.

(f) If A is the matrix of orthogonal projection onto a line, then $A^2 \neq A$.

Never. For all scalars k,

$$\text{proj}_{\vec{a}}(k\vec{a}) = \frac{(k\vec{a})\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a} = k\left(\frac{\vec{a}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = k\vec{a}$$

$$\Rightarrow \underbrace{\text{proj}_{\vec{a}}(\text{proj}_{\vec{a}}(\vec{x}))}_{= A^2\vec{x}} = \text{proj}_{\vec{a}}\left[\left(\frac{\vec{x}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a}\right] = \left(\frac{\vec{x}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = \underbrace{\text{proj}_{\vec{a}}(\vec{x})}_{= A\vec{x}}$$

So $A^2 = A$.

3. (a) Find the inverse of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 2 & 3 & -5 \\ -1 & -1 & 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 2 & 3 & -5 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathrm{I}) \leftrightarrow (\mathrm{III}) \quad \left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 0 & 0 & 1 \\ 2 & 3 & -5 & 0 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} (\mathrm{II}) + 2(\mathrm{I}) \\ (\mathrm{III}) + 2(\mathrm{I}) \end{array} \quad \left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & -2 & 3 & 1 & 0 & 2 \end{array}\right)$$

$$\begin{array}{l} (\mathrm{I}) + (\mathrm{II}) \\ (\mathrm{III}) + 2(\mathrm{II}) \end{array} \quad \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 6 \end{array}\right)$$

$$\begin{array}{l} (\mathrm{I}) - (\mathrm{III}) \\ (\mathrm{II}) + (\mathrm{III}) \end{array} \quad \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & -1 & -3 \\ 0 & 1 & 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 1 & 2 & 6 \end{array}\right)$$

$$(-1) \times (\mathrm{I}) \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 1 & 2 & 6 \end{array}\right)$$

So $\begin{pmatrix} 2 & 0 & -1 \\ 2 & 3 & -5 \\ -1 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 8 \\ 1 & 2 & 6 \end{pmatrix}$

(b) Express $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$

$$x\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + z\begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{pmatrix} 2 & 0 & -1 \\ 2 & 3 & -5 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 2 & 3 & -5 \\ -1 & -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 8 \\ 1 & 2 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 1 + 3 \\ 1 + 3 + 8 \\ 1 + 2 + 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 9 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + 12\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + 9\begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$

4. Find the matrices of the linear transformations $T \circ S$ and $S \circ T$, where $S : \mathbb{R}^3 \to \mathbb{R}^2$ projects each vector onto its first two coordinates and $T : \mathbb{R}^2 \to \mathbb{R}^3$ embeds $\mathbb{R}^2$ into the (x,z)-plane:

$$S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$$

Let A = matrix of S & B = matrix of T

$$S\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad S\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So the matrix of $T \circ S$ is

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}}$$

and the matrix of $S \circ T$ is

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}}$$